

Engrs. Library

TG

270

.C44

GRAPHICAL DETERMINATION OF FORCES
IN
ENGINEERING STRUCTURES.



1590

GRAPHICAL
DETERMINATION OF FORCES
IN
ENGINEERING STRUCTURES.



BY
JAMES B. CHALMERS, C.E.

London:
MACMILLAN AND CO.
1881.

The rights of Translation and Reproduction are reserved.

LONDON :
R. CLAY, SONS, AND TAYLOR,
BREAD STREET HILL.

TO

SIR ANDREW CLARKE, R.E., C.B., K.C.M.G.,

This work is dedicated,

BY PERMISSION,

AS A TOKEN OF ESTEEM BY HIS OBLIGED SERVANT,

THE AUTHOR.

LONDON,

March, 1881.

PREFACE.

THE following Treatise is confined to the more valuable applications of the graphical method, the creation of Professor Culmann of Zürich. Were a preference for this method once established, it might then in England, as now in Germany and Italy, be built up into a systematic discipline, remarkably fitted to exercise the intellectual powers. Beginning with *Projective Geometry* as founded by Poncelet, followed by *Geometric Statics*, as taught by Möbius, the *Calcul par le Trait*, of Cousinery, the properties of the *Funicular Polygon* of Varignon, so remarkably extended and employed by Culmann, and their application to the *Elastic Line* by Mohr, we should have a course of Engineering Mechanics, so invigorating to the mind, that our students, having undergone its discipline, would feel themselves men, well prepared for work, capable of appreciating the conditions, and reasoning upon the data; of a large class of practical questions to which they might require to address themselves.

From its birthplace in Switzerland (1860), this method has passed into Germany, Austria, Italy, Russia and Denmark, where, after Culmann, Wilhelm Ritter, Cremona, Favaro, and many others have communicated their enthusiasm to their pupils, through whose superior discipline their respective countries may soon rival in Engineering fame that of the

country which we fondly regard as its cradle, and as its most able instructress.

The reason, we apprehend, lies, first of all, in its geometric character. The designs of an Engineer are geometric conceptions, his structures are geometric forms, within which forces statically combined act along geometric lines, so that it is natural that he strive to follow a train of geometric thought.

For the progress of Statics and Geometry act and react upon each other. "Between Statics and Geometry an intimate connection exists. Not only does the first of these two sciences absolutely require the help of the other, but, for the given help, Statics furnishes Geometry with new Theorems, which, not seldom, again can be employed to the advantage of Statics." ¹

Geometric methods, further, possess a much higher value than analytical in expanding the intellectual powers. "Analysis excels, it is true, in arranging problems in equations, in disengaging, by a series of transformations, the combinations of symbols, which give the key to the question propounded, but its very perfection as a means of research neutralises its efficacy as a means of intellectual culture. Leading to the result by a procedure in some manner mechanical, the mind loses sight of the realities upon which it operates, it advances along a labyrinth of formulæ, intent only that it lose not the conducting thread, obliged to be more confiding as the darkness becomes more profound, and nearly always unconscious of the path along which it has travelled. On the other hand, according to the extremely just observation of Poincot, it is not rare that the results to which analysis conducts, remain concealed under the generality of algebraic symbols, so far as to appear even with less clearness in the solution than in the enunciation.

¹ Möbius, *Lehrbuch der Statik*. Leipzig, 1837.

Geometry proceeds wholly otherwise; she presents the propositions under a sensible form, she removes the train of auxiliaries which hide them from our view, she puts in evidence the transformations which each problem undergoes, and when the solution appears we now perceive the truth under a form the most simple and the most attractive.”¹ And we may add what our own countryman, Robert Simson (1687—1768), was wont to say of analysis, “A mere mechanical knack, in which we proceed without ideas of any kind, and retain a result without meaning, and therefore without any conviction of its truth.” Without, however, endorsing such an extreme view, we would say, in order that analysis may be rendered useful as an intellectual exercise, there must be constant attention given, at every possible stage, to its geometrical or physical interpretation.

The success of this method is, however, entirely due to modern geometric methods. The aim which modern geometers have constantly had in view can be best stated by one of their number: “In the actual state of the mathematical sciences, the only means whereby to prevent their domain from becoming too vast for our intelligence is, more and more to generalise the theorems which these sciences embrace, in order that a small number of truths, general and prolific, may, in the head of man, be the abridged expressions of particular facts.”²

Of these prolific generalisations the earliest were Desargues’ *Involution of Six Points* (1597—1661); Pascal’s *Hexagon*, 1640 (1623—1662); Carnot’s *Correlation des Figures de Géométrie*, 1801; *Transversals*, 1806 (1753—1823).

¹ Favaro, *Leçons de Statique Graphique, première partie, Géométrie de Position*. Paris, 1879.

² Dupin, *Développements de Géométrie*.

Here we come to the period when modern geometry presents itself. Poncelet (1813), "in the prisons of Russia, deprived of all species of books, and of all assistance, distracted by the misfortunes of my country and of my own," laid the foundation of his work "On the Central Projection of Figures in general, and of Conic Sections in particular, &c."¹ which entitle him to be called the creator of "Modern Geometry."

"In it were taught the principles that theorems concerning infinitely distant points may be extended to finite points on a right line, that theorems concerning systems of circles may be extended to conics having two points common, and that theorems concerning imaginary points and lines may be extended to real points and lines."² This he has been enabled to do by the "principle of continuity," and his method, we conceive, is best adapted for initiating those whose business is to apply geometry into fundamental conceptions of modern methods and into right appreciation of their value and power. This fitness was present to the mind of Poncelet himself, and we find on the title-page of his great work, "*Ouvrage utile à ceux qui s'occupent des applications de la géométrie descriptive et d'opérations géométriques sur le terrain;*" and in the Introduction, "*Les applications de la nouvelle géométrie, deviendront plus multipliées, plus nécessaires au grand nombre de ceux qui se vouent aux arts. Peu à peu aussi les connaissances algébriques deviendront moins indispensable, et la science, réduite à ce qu'elle doit être, à ce qu'elle devrait être déjà, sera ainsi mise à la portée de cette classe d'hommes qui n'a que des moments fort rares à y consacrer.*"

Poncelet was followed in Germany by Möbius (1790—1858), who greatly increased its power, in his *Barycentrische Calcul*,

¹ *Traité des Propriétés Projectives des Figures*, 1822.

² Salmon's *Conic Sections*, p. 302, Fifth edition. London, 1869.

(1827), and many later papers; and by Steiner (1796—1858); and in France by the brilliant Chasles (1793—1880), in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie, &c.* (1837), in his *Traité de géométrie supérieure* (1852), and many other contributions; and by the unique work of Von Staudt (1847), with which Culmann supposes his pupils to be acquainted. It is little fitted, however, for a general discipline in the “conflict of studies” and the hurry of life in France and England.

Möbius may likewise be said to have created Geometric Statics in his *Lehrbuch der Statik* (1837), converting it from a number of isolated propositions into a distinct branch of science.

Meanwhile attempts were not wanting to apply the new developments to the wants of the Engineer. Poncelet, according to Culmann,¹ employed the funicular polygon in the determination geometrically of the centre of gravity.

Lamé and Clapeyron (1826), give two “Mémoires sur la Construction des Polygones Funiculaires,” in *Journal des Voies de Communication*, St. Petersburg (Dec. 1826, Jan. 1827). Poncelet gives in *Mémorial de l'Officier de Génie* (1840), a “Mémoire sur la Stabilité des Revêtements et de leurs Fondations.”

Cousinery in his *Calcul par le Trait*, devotes the fourth part to “Applications des Procédés du Calcul Graphique à la Solution des divers Problèmes de Stabilité,” the contents of which are generally the conditions of stability of revetment walls and the equilibrium of arches.

Méry, in *Annales des Ponts et Chaussées* (1840), has exposed

¹ Culmann, *Die Graphische Statik*, p. viii. Zürich, 1875.

the curve of pressure in an arch. Durand-Claye, in 1867 and 1868, likewise in these *Annals* has generalised and extended the methods.

Peaucellier, in *Mémorial de l'Officier de Génie* (1875), has given a most elegant solution of the stability of a stone arch in regard to friction between the voussoirs.

Rankine's "Equilibrium of Impressed Forces in a Polygonal Frame" (1858),¹ with Clerk-Maxwell's generalisation² (1864) of Taylor's discovery of reciprocal frame and force diagrams,³ were important English developments, traceable to the teaching of Möbius.

These two developments require, however, to be supplemented by Culmann's method of obtaining the two reacting forces, in the case of ordinary frames, more especially when the impressed forces are not parallel. By themselves they remained comparatively unfruitful.

Since then, Cremona, Culmann, Maurice Levy, and Friedrich Steiner have succeeded in giving elegant statical demonstrations of that remarkable property of reciprocity.

Here then we see the reason for the success of Culmann upon the Continent; when (1860—1866) he applied Modern Geometry to Engineering problems, he could obtain an audience prepared to understand him, and in the widespread University and High School System of Germany were a class of men to whom the works of Poncelet, Möbius, Chasles, and even Staudt, were familiar, ready to adopt his method, everywhere receiving from

¹ Rankine, *Manual of Applied Mechanics*. London, 1858.

² "On Reciprocal Figures and Diagrams of Forces." Prof. J. Clerk-Maxwell, *Phil. Mag.* April, 1864.

³ "On the Practical Application of Reciprocal Figures to the Calculation of Strains on Framework." Prof. Fleeming Jenkins, *Trans. Roy. Soc. Edin.* vol. xxv. 1869.

their hands some valuable improvement or some half concealed property brought by them more prominently into the foreground. But it was otherwise in England. In 1852 a Fellow of Cambridge could write, "The principles of Modern Geometry have hitherto received little attention in this University,"¹ and even he is wholly analytic in his demonstrations; and the first systematic treatise on Modern Geometry appeared so late as 1863—1865.²

The mathematicians of England, in fact, may be said at two different times to have exchanged studies with those of the Continent. From Newton's *Principia* (1687), a line of mathematicians, Maclaurin, Halley, Robert Simson, and Matthew Stewart, followed the methods of ancient Greek geometry. Maclaurin employed it with consummate skill. Simson in the *Loci Plani of Apollonius restored* (1749), and in his *Porisms* (pub. 1776), recovered somewhat of the ancient geometry which was lost; and in Stewart's *General Theorems* (1748), a step in advance of the ancients had been gained, but there ends the line of geometric kings, for Leslie's *Geometric Analysis* (1821), is to some extent the modern calculus disguised. On the other hand, from the publication by Leibnitz of his *Calculus* (1684), the Continental mathematicians pursued with ardour discovery in that new world of science now within their horizon; the calculus becoming, in the hands of the Bernouillis (—1748), Euler (—1768), Clairaut (—1765), Lagrange (—1813), Laplace (—1825), an instrument of research of mighty power.

Then came an interchange. England finding herself distanced

¹ Gaskin, *Geometrical Construction of a Conic Section*. Cambridge, 1852.

² Townsend, *Modern Geometry of the Point, Line, and Circle*. Dublin, 1865.

in the race of mathematical discovery, and typically represented by three young men of Cambridge—Herschell, Babbage, and Peacock (1816)—urges the adoption of the notation of Leibnitz and renewed efforts in analysis. In this revival of the claims of analysis a gentle Scotchwoman bore a distinguished part. Mrs. Somerville's *Mechanism of the Heavens* was published in 1831. Turning again to the Continent. Whether it was that curious classic revival extending itself to geometry as well as to republican institutions and Grecian costume; or whether it was, as more likely, the influence of Monge (1746—1818),—Carnot, a General of Engineers under Napoleon, in his *Géométrie de Position* (1803), and in those other works which we have already signalised, prepared the way for Poncelet; and we have already remarked how entirely neglected at that period became geometry in England, and thus, however a few learned men may have kept abreast of the Continental geometers, their methods were neglected at the universities. To the best of our knowledge, Modern Geometry has only full recognition at present in one university, Trinity College, Dublin.

Thus there was no preparedness for Culmann's method. We regret to add the other reason which exists; the necessity of a scientific training of any kind for an Engineer has only partially been recognised. There are, no doubt, amongst us, a large number, who in earlier years have studied their Pratt, their Navier, their Moseley, or who in more recent years have become familiar with their Bresse and Rankine, have made themselves acquainted with Clapeyron's Theorem of the Three Moments, even a few to whom Lamé is not unknown, but those have done so without hope of reward, simply that they might be truthful men, knowing that which they as Engineers profess to know. But by how many are such surrounded, often

jostled, undistinguishable from them by the laity, committing blunders by rule of thumb, affecting to despise science, talking vaguely of their experience and of the practical, whence our public structures suffer in strength, elegance, and economy from vicious design, and our public works from defective method in their complete conception.

This opinion is, we are ashamed to say, widely held upon the Continent. Referring to suspension bridges, Collignon says,¹ "Several English suspension bridges owe their rigidity to an excess of material. Such a system has no economical merit—the only merit which a suspension bridge can possess." In reference to continuous girders, Weyrauch says,² "It is remarkable that the construction of continuous beams is most favourably received where Engineers can calculate, as in France and the South of Germany, and not there where the rule is that the bureau of the Engineer is conducted by people who sit upon two stools, and upon neither firmly." Here England is significantly omitted. The number of continuous beams on English railways might be counted on the fingers, and on the 9,000 miles of Indian railways, only upon the Madras line does a continuous beam occur, while in France they are employed over every considerable river, to the attainment of the greatest economy in the most expensive undertakings. Again, Culmann says,³ "But what is appropriate to the rich Englishman, who everywhere carries himself about with great consciousness, 'I am in possession of the iron, and do not require to trouble myself about statics,' is not so to the poor devils of the Continent. They must meditate and experiment, they

¹ Collignon, *Cours de Mécanique*. Première Partie. Paris, 1877.

² Weyrauch, *Allgemeine Theorie und Berechnung der Continuerlichen und Einfachen Träger*. Leipzig, 1873.

³ Culmann, *Graphische Statik*. Zurich, 1866.

must lay off and estimate many lines for each one projected, in order to find the cheapest, and they must make out many projects for every intended bridge, in order not to waste the smallest quantity of material, employing only so much as is absolutely indispensable."

We trust that this work will conduce in some small degree to the spread of a more perfect appreciation of careful design in structures. We have attempted, within moderate compass, to give it an immediately practical value, and, from first to last, theory has been subordinated to practical application. The systematic discipline, as sketched at the commencement of this preface, has not been attempted. Statics, as commonly taught, and the Summation of Elementary Integrals, are supposed to have been previously studied. We have, however, added a chapter on Projective Geometry, carried so far as is necessary to the demonstrations of the Treatise, mainly following Poncelet as the most instructive form in which we could present it. A student should by no means be satisfied with this, but we hope that he will find Poncelet's principle of the infinitely distant point and line so treated as to aid him when he proceeds to any treatise on Modern Geometry. For modern geometrical writers are so chary of figures that the way of a student is rendered by no means so pleasant and easy as it would otherwise be. We would recommend to our readers who wish to proceed further, to begin with the French translation of Cremona's *Projective Geometry*, from which they may proceed to the writings of Chasles. For a right understanding of Reciprocal Figures in the Statics, we would recommend a brochure by Friedrich Steiner, *Die Graphische Zusammensetzung der Kräfte*, Wien, 1876. How much we are indebted to Culmann, it would be difficult to over-estimate. We claim nothing original, but we have made it our

endeavour, above everything else, to be clear, and hope that we have fairly succeeded. We give, for the first time, the geometrical construction of Durand-Claye's hyperbolas; the demonstrations in art. 81. The coincidence of the centre of gravity of a trapezium with a point of the antipolar to the intersection of its uniting lines as pole is, no doubt, well known, although we cannot remember to have seen it referred to. The union of the funicular polygon with Durand-Claye's method for finding the lines of pressure in the arch and with the formation of reciprocal frame and force diagrams is more clearly exhibited in both cases than it has been before. We believe the graphical demonstration in art. 218 is given for the first time.

No one could be more alive to the imperfections of our work than we are ourselves, but we have believed that we supply a want, and have unfolded to the English student a powerful method, only to be learned from the lectures of Continental professors, or from Culmann's abstruse work, and from papers scattered up and down German scientific periodicals and brochures. If we may be permitted to make an apology, we would say that it was written during leisure hours in the midst of the ordinary pursuits of the Engineer.

Upon the value of this method, of its remarkable facility, practical accuracy, and invariable self-verification, we will quote the words of M. Levy.¹ "Elle met à la disposition de tous, pour tenir lieu des savants et laborieux calculs auxquels se livrent encore journellement nos ingénieurs, des procédés simples et expéditifs. Ces procédés ont de plus le précieux avantage de porter toujours en eux-mêmes le principe de leur vérification, de telle sorte que s'ils peuvent, comme toutes les méthodes graphiques, laisser un doute sur une fraction décimale,

¹ Levy, *Statique Graphique*. Paris, 1874.

chose très-indifférente en ce genre d'applications, ils sont en revanche exempts de ces chances d'erreurs grossières que comportent les longues opérations arithmétiques et les formules algébrique, où rien ne parle aux yeux."

In conclusion, we hope that this work will recommend itself to Engineers in the exercise of their profession, and be found to serve as a convenient text-book in Engineering classes.

CONTENTS.

CHAPTER I.

PARALLEL FORCES IN ONE PLANE.

Section I.—Introductory.

| Art. | Page |
|---|------|
| 1. Preliminary Problem, Case I. | 1 |
| 2. Definition of Force and Cord Polygon | 3 |
| 3. Displacement of Pole of Force Polygon | 3 |
| 4. Preliminary Problem, Case II. | 3 |
| 5. Preliminary Problem, Case III. | 4 |
| 6. Scales of Force and Cord Polygons | 4 |
| 7. Rule in regard to Scales of Force and Cord Polygons . . . | 5 |
| 8. Inversion of Force and Cord Polygon | 6 |
| 9. Summation of the Intercepts $av : h$ Independent of the Order of their Seizure | 6 |
| 10. Axial Line of Two Cord Polygons to the same Pairs of Factors but differing in the Order of Seizure of One of the Pairs . . | 6 |

Section II.—Line of Action of Resultant of Parallel Forces in One Plane, and Centre of Gravity of a Lamina.

| | |
|--|----|
| 11. Line in which $\Sigma av : h = 0$ | 8 |
| 12. Line of Action of Resultant of Parallel Forces | 8 |
| 13. Centre of Gravity of a Lamina | 10 |
| 14. Centre of Gravity of a Lamina having an Axis of Symmetry . | 10 |
| 15. Representation of Areas by Lines | 10 |
| 16. Verticals through Centres of Gravity of Number of Wheels of Locomotives and Tenders | 13 |
| 17. Centre of Gravity of a Trapezium | 13 |
| 18. Centre of Gravity of an Irregular Quadrilateral | 14 |
| 19. Centre of Gravity of a Lamina having no Axis of Symmetry. | 15 |

Section III.—Equilibrium in an Ideal Beam between Parallel Forces.

| Art. | Page |
|---|------|
| 20. Equilibrium between Three Parallel Forces | 16 |
| 21. Equilibrium between any number of Parallel Forces | 19 |

Section IV.—Equilibrium in the Cord Polygon formed by Parallel Forces.

| | |
|--|----|
| 22. Equilibrium of the Various Members of the Cord Polygon of Parallel Forces | 20 |
| 23. Alteration of Sign in Tensions of Links of Cord Polygon by Inversion of its Position | 22 |
| 24. Horizontal Components of the Tensions in all the Links of Cord Polygon have the same Value | 22 |

Section V.—Action of Parallel Forces in one Plane in a Beam.

| | |
|--|----|
| 25. Shearing Force and Bending Moment | 22 |
| 26. Reasons for the Names | 24 |
| 27. Measure of Shearing Force and Bending Moment | 24 |
| 28. Shearing Force and Bending Moment in a Beam fixed at One End | 24 |
| 29. Compound Nature of the Measure of Bending Moment | 25 |
| 30. Moment of Encastment of a Beam | 25 |
| 31. Shearing Force and Bending Moment of a Beam supported at Both Ends | 25 |
| 32. Scale to Bending Moment | 27 |
| 33. Preparation of a Diagram of Bending Moments | 27 |

Section VI.—Loci of Maximum Bending Moment in Beam under Parallel Forces and between two Points of Support.

| | |
|---|----|
| 34. Locus of Greatest Bending Moment for any Given System of Weights, in One Given Position on a Beam | 28 |
| 35. Positions of Maximum of Greatest Bending Moment, arising from a System of Weights | 29 |

Section VII.—Further consideration of the Two Supporting Forces of a Beam.

| | |
|---|----|
| 36. Reactions of the Two Points of Support of a Beam | 30 |
| 37. Identity of the Ordinary and Graphical Methods of Finding the Reactions in a Beam | 31 |

Section VIII.—Additional Forces Brought or Taken Off a Beam considered.

| | |
|--|----|
| 38. Interpolation of an Additional Weight at any Point of a Beam | 31 |
|--|----|

CONTENTS.

xxi

| Art. | Page |
|--|------|
| 39. Forming a Cord Polygon with Additional Interpolated Forces | 31 |
| 40. Interpolating the Resultants of Additional Series of Forces in a Cord Polygon | 34 |
| 41. Alterations in Value of Bending Moment and Shearing Force arising from Taking Weights off the Beam | 34 |
| 42. Value and Line of Action of Resultant of Forces on One Side of a given Point in a Beam | 36 |
| 43. Negative Bending Moment | 37 |
| <i>Section IX.—Turning Bending Moment Couples.</i> | |
| 44. Application to an Arched Beam | 37 |
| 45. Application to a Warehouse Crane | 38 |
| <i>Section X.—The Older Graphical Methods of Obtaining Bending Moment and Shearing Force.</i> | |
| 46. Shearing Force Treated by the Older Graphical Method | 40 |
| 47. Bending Moment Treated by the Older Graphical Method | 42 |
| 48. Change of Sign of Shearing Force in a Beam under Traveling Loads, taking the Weight of the Beam into Account | 43 |
| 49. Geometrical Determination of the Curves of this Section | 44 |

CHAPTER II.

OPEN FRAME WORK.

Section I.—Extension of Force and Cord Polygon Theorems to Forces distributed in any manner in a Plane.

| | |
|---|----|
| 50. Formation of Force and Cord Polygon to Forces distributed in any manner in a Plane | 45 |
| 51. Culmann's Theorem (51-53). Axial Line and Reciprocity of Figure between a Combined Pair of Force and a Combined Pair of Cord Polygons, belonging respectively to the same Forces with One Displaced Force | 45 |
| 52. Elementary Demonstration of Reciprocal Quadrilaterals | 48 |
| 53. Axial Line and Reciprocity between the Combined Pair of Force and Combined Pair of Cord Polygons to the same Forces, Pole being displaced | 51 |
| 54. Resultant and Line of Action of Resultant of Forces in a Plane | 51 |

Section II.—Application of preceding Force and Cord Polygon. Theorems to Determination of Tensions or Stresses in the Bars of Framework.

| Art. | Page |
|--|------|
| 55. Condition of Indeformability in Framework | 51 |
| 56. Example of Irregular and Simple Framing | 51 |
| 57. Definition of Frame Diagram and Force Diagram | 54 |
| 58. Method of Sections, assisted by Line of Action of Resultant of Forces on One Side of Section | 54 |
| 59. Method of Sections (continued) | 56 |
| 60. Variations in Form of Force Diagrams | 59 |
| 61. Method of Sections applied to Forces acting upon the Joints of a Frame in any direction | 59 |
| 62. Cases in which Force and Frame Diagrams are Similar Figures | 61 |
| 63. Method of obtaining Force Diagram by Method of Sections, independently of Line of Action of Resultant of Forces on One Side of Section | 61 |
| 64. Method of Sections applied to the Construction of Partial Force Diagrams in regard to Travelling Load | 64 |
| 65. Graphical Solution of Analytical Expression for Stress in the Bars of a Frame | 68 |

Section III.—Simplifications in Force Diagrams.

| | |
|--|----|
| 66. Enumeration of Simplifications | 69 |
| 67. Force Diagrams of a Parallel Boomed Girder | 69 |
| 68. Partial Force Diagrams in a Parallel Boomed Girder | 70 |
| 69. Force Diagrams of Roof Framing | 71 |

Section IV.—Clerk-Maxwell's Theorem.—Reciprocal Force Diagrams.

| | |
|---|----|
| 70. Conditions necessary in order to obtain a completely Reciprocal Figure as Force Diagram to a given Frame | 77 |
| 71. Examples of Formation of Reciprocal Diagrams to Frames of perfectly General Form with Forces in any direction | 78 |
| 72. Application of Reciprocal Diagrams to Wind-pressure on Roof | 82 |
| 73. Examples of Application of preceding Methods | 82 |

Section V.—Herr Heuser's Problem and its Application to Hinged Archwork.

| | |
|---|----|
| 74. Problem. To carry a Cord Polygon to given Forces through Three given Points | 85 |
| 75. Application to Hinged Archwork | 87 |

Section VI.—Compression and Tension in Bars.

| Art. | Page |
|--|------|
| 76. Analysis of the Conditions giving Compression or Tension in Bars | 88 |
| 77. Rule by which to Discover the Sign of Stress in a Bar . . . | 89 |

Section VII.—Limit to which a Travelling Load must extend over an Open Framework in order to give a Maximum Stress in any given Bar.

| | |
|--|----|
| 78. Limit for a Girder of General Form | 89 |
| 79. Limit in a Parallel Boomed Girder | 91 |
| 80. Limits in a Hinged Archwork | 92 |
| 81. Partial Diagrams of Maximum Stress in Hinged Archwork . | 94 |
| 82. Combined Series of Partial Diagrams giving Maximum Stresses in Hinged Archwork | 97 |
| 83. Corresponding Maxima and Minima in a Symmetrical Parabolic Arch | 98 |

Section VIII.—Method of Determining the Stresses in the Parts of a Girder directly from the Shearing Force and Bending Moments.

| | |
|---|-----|
| 84. Example in a Parallel Boomed Girder | 100 |
|---|-----|

Section IX.—Forms of Girders satisfying given Conditions.

| | |
|--|-----|
| 85. The possibility of satisfying given Conditions | 102 |
| 86. The Parabolic Girder | 102 |
| 87. The Schwedler or Hyperbolic Girder | 104 |
| 88. Pauline Girder, or Bow and String Girder with equal Stress throughout the Boom | 105 |
| 88(a). Compound Girders | 106 |
| 88(b). Position of Roadway | 106 |

Section X.—Problems in Cranes combining several of the Operations of the preceding Two Chapters.

| | |
|---|-----|
| 89. Pivot Reaction and Force Diagram of a Fairbairn Crane . . | 106 |
| 90. Bending Moments and Stresses in Common Crane | 107 |

CHAPTER III.

THE BEAM.

Section I.—General View of the Forces Applied and Resisting in a Beam.

| | |
|---|-----|
| 91. Definition of a Beam | 110 |
| 92. The Exterior Forces applied to a Beam | 110 |
| 93. Resisting Stresses in a Beam | 111 |

| Art. | Page |
|--|------|
| 94. Nomenclature of the Stresses in a Beam | 111 |
| 95. Hypothesis in regard to the Stresses in a Beam | 112 |
| <i>Section II.—Beam with Normal Forces only in the Plane of Symmetry.</i> | |
| 96. Conditions necessary to Equilibrium between Forces and Stresses | 112 |
| 97. Radius of Gyration | 114 |
| 98. Fundamental Formula for Stress in Outer Filament of Beam . | 114 |
| 99. Comparison between Formula for Stress and for Stress in Booms of Open Girder | 114 |
| 100. Modulus of Elasticity | 115 |
| 101. Radius of Curvature of a Beam, and Differential Equation of Deformed Neutral Axis | 116 |
| 102. Geometrical Signification of $\frac{d^2z}{dy^2}$ | 118 |
| 103. Ratio of Distances of Upper Arched Boom and Lower Arched Boom from the Common Horizontal Chord in the Pauline Girder | 120 |
| 104. Graphical Construction of Moments and Products of Inertia, in the Case of a System of Heavy Particles | 120 |
| 105. Theorem. Moments of Inertia of a Lamina about any Axis . | 122 |
| 106. Theorem. Moment of Inertia, and Radius of Gyration of a Rectangular Lamina | 124 |
| 107. Graphical Construction of the Moments of Inertia of a Rectangular Lamina, around any Axis parallel to one of its Sides | 124 |
| 108. Graphical Construction of the Combined Moments of Inertia and Combined Radius of Gyration of several Rectangular Laminæ | 126 |
| 109. Graphical Construction of the Moments of Inertia and Radii of Gyration of the Cross Section of a Beam | 129 |
| 110. Apportioning the Area of a Beam to the Bending Moment . | 130 |
| 111. Moments of Inertia and Radii of Gyration in a Rail | 130 |
| ✓ 112. Distribution of Shearing Stress on the Longitudinal Section of a Beam | 131 |
| ✓ 113. Graphical Construction and Representation of the Distribution of Shearing Stress in a Beam | 134 |
| 114. Recapitulation of Formulæ | 137 |
| <i>Section III.—Beam with Oblique Forces P in the Plane of the Vertical Axis of Symmetry.</i> | |
| 115. Effect of Oblique Force in Plane of Symmetry in removing the Neutral Axis | 138 |

| Art. | Page |
|---|------|
| 116. Point of Action in Cross Section of Resultant of Stresses . . | 140 |
| 117. Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are Pole and Antipolar | 140 |
| 118. Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are interchangeable | 141 |
| 119. Point of Action of Forces when Neutral Axis is removed to Edge of Cross Section | 141 |
| 120. Finding the Neutral Axis at a given Section due to a given Oblique Force | 143 |
| 121. The Neutral Axis in an Arch, or Retaining Wall | 143 |
| 122-124. <i>Geometrical Construction of Durand-Claye's Hyperbolas.</i> | |
| 122. Problem. To Construct the Valves Y of Stress in a Rectangular Cross Section for every point of its application under Constant Stress of Outside Lamina | 144 |
| 123. Extension of preceding Problem to Symmetrical Beam . . . | 146 |
| 124. The Locus of $Y : 2ab$ is an Equilateral Hyperbola | 148 |
| <i>Section IV.—Case of an Unsymmetrical Beam, or of a Symmetrical Beam unsymmetrically placed, the Plane in which the Forces act going through the Centre of Gravity of the Beam.</i> | |
| 125. Condition of Equilibrium in Case in which $Zx = 0$ | 148 |
| 126. Transferring the Axes around which the Moments and Products of Inertia are taken, parallel to themselves | 149 |
| 127. The Ellipse of Inertia and its Construction | 151 |
| 128. When the Vertical Component Z of the Applied Forces only exists, the Neutral Axis is the Diameter of the Ellipse of Inertia conjugate to that Diameter which, in the Cross Section, is the Trace of the Plane going through Z and the Centres of Gravity of the Cross Sections of the Beam | 155 |
| 129. If the Line of Action of the Resultant of Applied Forces passes through the Point Z , in the Trace of the Vertical Plane through ZG in which the Applied Force lies, then the Neutral Axis is removed to the Antipole of Z and lies parallel to the Conjugate Diameter of ZG , and conversely . | 155 |
| 130. Core or Heart of a Lamina | 155 |
| <i>Section V.—Problems in connection with Moments and Products of Inertia.</i> | |
| 131. Having the Central Ellipse O of any Lamina, to Construct the Moments and Products of Inertia of that Lamina around any other given Axes | 159 |

| Art. | Page |
|--|------|
| 132. Employment of the Methods of last article to an Unsymmetrical Angle Iron | 162 |
| 133. Construction of Ellipse of Inertia around any Point of a Lamina, having given the Central Ellipse | 167 |
| <i>Section VI.—Construction of the Central Ellipse and Core of several Laminae, having Elementary Plane Forms.</i> | |
| 134. Central Ellipse and Core of a Parallelogram | 168 |
| 135. Central Ellipse and Core of a Triangle | 170 |
| 136. Central Ellipse and Core of a Trapezium | 172 |
| <i>Section VII.—Planes of Greatest and Least Stresses in a Beam.</i> | |
| 137. Force Polygon of Stresses upon an Elementary Solid in a Beam | 176 |
| 138. Planes of Maxima and Minima Stresses in the Elementary Solid | 178 |
| 139. Curves of Maximum and Minimum Stresses | 179 |
| 140. Tractories of Maxima Stresses upon the Longitudinal Section of the Beam | 180 |
| 141. Construction of the Lines u of Maxima Stresses on a Beam | 182 |

CHAPTER IV.

CONTINUOUS BEAM.

| | |
|---|-----|
| 142. Preliminary Remarks | 197 |
| 143. Professor Mohr's Problem. Geometrical Interpretation of the Equation to the Elastic Line | 199 |
| 144. Construction of Elastic Line Tangent Envelope of a Beam lying freely upon Two Supports with Forces only between them | 204 |
| 145. Construction of Elastic Line Tangent Envelope of an Unloaded Span, with Moments over the Supports arising from Forces Exterior to the Span | 207 |
| 146. Construction of Elastic Line Tangent Envelope for Two Adjacent and Unloaded Spans | 210 |
| 147. Dependence of the Inflexion Points of a Series of Unloaded Spans upon the First Inflexion Point | 211 |
| 148. A Geometrical Theorem | 212 |
| 149. Construction of the Inflexion Points of a given Continuous Beam | 213 |
| 150. Given a Weightless Beam, having One Span only Loaded, with a Concentrated Weight P , to find the Bending Moments over the Supports, and the Elastic Envelope giving the Tangents over the Supports | 215 |

| Art. | Page |
|---|------|
| 151. Remarks upon Choice of Constants | 218 |
| 152. Moments over all the Supports of a Weightless Beam having only One Span Loaded, and its Elastic Line | 221 |
| 153. Example of a Continuous Beam with One Concentrated Load in each Span | 221 |
| 154. Extension of Problem to a Number of Concentrated Loads in any Spans | 224 |
| 155. Simplifications introduced from Uniform Loading | 224 |
| 156. Example of Continuous Beam with Uniform Loading . . . | 225 |
| 157. Uniform Distribution of Pressure over Part of a Span . . . | 226 |
| 158. Scheme of Loadings for Determination of Maximum Positive and Negative Moments | 229 |
| 159. Ratio of Ordinates of Constructed Elastic Line to those of the True Elastic Line | 230 |
| 160. Shearing Force | 231 |
| 161. Most Unfavourable Pressures | 232 |
| 162. Recapitulation of most Unfavourable Conditions of Loading. | 235 |
| 163. Maximum Value of Reaction | 236 |
| 164. Structural Weight of Beam | 236 |
| 165. Influence of the Height of the Points of Support | 236 |
| 166. Case of Continuous Beam with Varying Longitudinal Force . | 236 |
| 167. Loaded Beam Inbuilt at <i>A</i> and <i>B</i> | 237 |
| 168. Equation of Elastic Line to a Beam of Variable Cross Section | 237 |
| 169. Elastic Line of a Loaded Beam, inbuilt at one end, free at the other, and having a Variable Cross Section | 237 |
| 170. Influence of Variable Cross Section upon the Stresses . . . | 238 |
| 171. Determination of the <i>v</i> and <i>u</i> Lines in a Continuous Beam of Variable Cross Section | 239 |
| 172. Formation of Elastic Line Envelope of an Unloaded Span . | 241 |
| 173. Maxima of Shearing Force and Bending Moment in any Span of a Continuous Beam under Concentrated Loads | 244 |

CHAPTER V.

THE ARCH.

*Section I.—On Pressures Communicated through Bodies in Contact
with each other.*

| | |
|---|-----|
| 174. Value of Friction and of Stability in reference to Sliding . . | 246 |
| 175. Plane of Repose | 247 |
| 176. Stability in Reference to Overturning | 248 |
| 177. Line of Pressure in a Series of Bodies in Contact | 249 |
| 178. Moseley's Line of Resistance | 249 |

Section II.—The Arch considered.

| Art. | Page |
|---|------|
| 179. Lines of Pressure and Resistance in an Arch | 250 |
| 180. Recovering a Point in the Line of Resistance from the Line of Pressures | 250 |
| 181. The True Line of Pressures | 252 |
| 182. M. Durand-Claye's Method of Finding the Admissible Lines of Pressure in an Arch | 252 |
| 183. Application of Heuser's Problem to Archwork | 256 |
| 184. Curvature of Arch | 256 |
| 185. Finding the Most Axial Line of Pressures | 258 |
| 186. Bending and Radial Shearing Stress | 258 |
| 187. Stresses in Tables of Beam | 260 |
| 188. Stresses in Bars of Open Arch Beam | 260 |
| 189. Critical Sections of an Arch | 262 |
| 190. Conditions necessary to Equilibrium in a Pointed Arch . . . | 262 |

*Section III.—The Elastic Arch. Professor Culmann's Method.—Part**A. The Equations of the Elastic Arch.*

| | |
|--|-----|
| 191. Lemma | 263 |
| 192. Lemma | 264 |
| 193. Differential Equation of Form Alteration | 265 |
| 194. Turning Power of Elementary Form Alteration | 266 |
| 195. Turning Power referred to Coordinate Axes | 266 |
| 196. Summation of Elementary Turnings | 266 |
| 197. Hypothesis of a Single Force U Equilibrating the Whole Turning | 266 |
| 198. Treating all $\frac{ds}{EI}$ as Forces, Antipole V of Line of Action of U , in reference to the Central Ellipse of all $\frac{ds}{EI}$ | 267 |
| 199. Value of Single Force U resisting Turning | 269 |
| 200. Multiples of Perpendiculars upon U from Centre of Gravity G , and from the Antipoles Y and Z of y and z | 269 |

*Section III.—Part B. Graphical Construction of the Lines of Action
and Values of Reacting Forces.*

| | |
|---|-----|
| 201. Preliminary Problem | 270 |
| 202. Modification of the Form of the Increment $\frac{ds}{EI}$ | 271 |
| 203. Equating Sum of Moments in Bow with Moments of U | 271 |
| 204. Formation of Force Polygon having $\Sigma \Delta y$ for its Line of Weights | 272 |

| Art. | Page |
|--|------|
| 205. Formation of First Cord Polygon | 273 |
| 206. Vertical through Centre of Gravity of $\Sigma \frac{\Delta s}{s_i}$ | 274 |
| 207. Horizontal through Centre of Gravity of $\Sigma \frac{\Delta s}{s_i}$ | 274 |
| 208. Formation of other Force and Cord Polygons | 275 |
| 209. Vertical through Antipole Z of z Axis | 275 |
| 210. Formation of other Force and Cord Polygons | 276 |
| 211. Formation of another Cord Polygon | 276 |
| 212. Axes of the Central Ellipse | 277 |
| 213. Direction and Value of Force U | 277 |
| 214. Envelope of the U Lines | 278 |
| 215. Total Reaction \tilde{U} at One of the Springings arising from a Force P | 278 |
| 216. Relation between the u Intercepts | 281 |
| 217. Reaction Intersection Curve, Envelope of Force U and Envelopes of the Reactions \tilde{U} and \tilde{U}' | 282 |
| 218. Interpolating U , \tilde{U} , \tilde{U}' Lines | 282 |
| 219. Lines of Pressure in the Arch | 282 |
| 220. Forces Generated by Expansion from Increase of Temperature | 282 |
| 221. Arch Hinged at the Springings A and B | 283 |
| 222. Practical Construction of Arches Hinged at Springing . . . | 284 |
| 223. Unfavourable Conditions of Loading | 284 |
| <i>Section IV.—Arch Centring.</i> | |
| 224. Forces resisted by an Arch Centring | 285 |

CHAPTER VI.

THE SUSPENSION BRIDGE AND AUXILIARY GIRDERS.

Section I.—Suspension Bridge.

| | |
|--|-----|
| 225. Curve assumed by Main Chains of Suspension Bridge . . . | 287 |
| 226. Method of Obtaining Curve of Main Chains | 287 |
| 227. Anchoring the Chain | 288 |
| 228. Suspension Bridge with Oblique Suspending Rods | 291 |
| 229. Anchoring by means of Vaulted Masonry behind the Pier . . | 291 |
| 230. Anchoring in Rock | 291 |

Section II.—Auxiliary Girders of Suspension Bridge.

| | |
|--|-----|
| 231. Reasons why Suspension Bridges require to be Stiffened . . | 291 |
| 232. Auxiliary Girders employed to Stiffen Suspension Bridge . . | 292 |

| Art. | Page |
|--|------|
| 233. Points of Support of Auxiliary Girder | 292 |
| 234. Reactions at Points of Support of an Auxiliary Girder and Cord Polygon of Distributed Pressures | 292 |
| 235. Construction of Reactions of Auxiliary Girder at Points of Support. Cord Polygon of Chain Reactions, and Cord Polygon of Partial Load | 293 |
| 236. Measure of Bending Moments of Auxiliary Girder | 293 |
| 237. Most Unfavourable Disposition of Travelling Load | 294 |
| 238. Measure of Shearing Force in Front of Travelling Load | 294 |
| 239. Auxiliary Shrouds of Mr. Ordish | 294 |
| 240. Horizontal Main Chains | 295 |

CHAPTER VII.

ON RETAINING WALLS.

Section I.—Statement of the Problem.

| | |
|---|-----|
| 241. (a) <i>Active Pressure of Earth upon a Retaining Wall</i> —Plane of Rupture | 296 |
| 242. The Forces in Equilibrium | 297 |
| 243. (b) <i>Passive Resistance of Earth</i> | 297 |
| 244. Plane of Rupture of Maximum Thrust | 298 |
| 245. Lines <i>P</i> representing Weight of Earth Prism for any Planes of Rupture | 299 |
| 246. Reduction of an Earth Prism with Irregular Terrain Line to a Triangular | 300 |
| 247. Transformation of a Loaded Earth Prism to an Appropriate Triangle | 300 |
| 248. Turning Force Polygon round through an Angle | 302 |
| 249. Hyperbola of Earth Pressure | 303 |
| 250. Construction of the Maximum Value of Earth Pressure | 304 |
| 251. Variations in Construction of Maximum Earth Pressure | 305 |
| 252. Lemma | 306 |
| 253. Triangle of Earth Pressure | 306 |
| 254. Pressure of Earth per Unity of Area at any Point <i>O</i> in a Retaining Wall | 307 |
| 255. Conditions Simplifying Construction | 310 |
| 256. To Reduce a Series of Pressures, <i>CD</i> , for a Series of Points, <i>O</i> , of a Wall, to One Base <i>e</i> | 311 |
| 257. Line of Action of Resultant of Pressure upon a Retaining Wall | 312 |

Section II.—Cohesion.

| Art. | Page |
|---|------|
| 258. Remarks | 313 |
| 259. Equilibrium of Earth Prism sustained by Cohesion alone . . | 313 |
| 260. Geometrical Construction of the Value of Cohesion | 314 |
| 261. Geometrical Construction of the Maximum Value of Cohesion | 315 |
| 262. Parabola of Cohesion | 315 |
| 263. Position of the Plane of Rupture for which the Influence of Cohesion in Diminishing the Earth Pressure is the Least . | 316 |
| 264. General Divergence of Planes of Rupture and Cohesion and Case of Coincidence | 317 |
| 265. Introducing the Influence of Cohesion into the Construction of the Earth Pressure | 318 |

Section III.—Designing Retaining Walls.

| | |
|--|-----|
| 266. Application of preceding Methods to one or more Series of Retaining Walls | 318 |
| 267. Efficacy of Pitching a Steep Slope | 323 |
| 268. Drawing in the Line of Resistance to a Retaining Wall, combined with Prof. Haeseler's Method of Representing the Earth Pressure | 323 |
| 269. Fluid Pressure behind a Retaining Wall | 324 |
| 270. Determination in a Wall Sustaining Fluid, of the Resultant Pressure on the Base | 325 |
| 271. Distribution of Pressure upon the Base of a Retaining Wall . | 326 |
| 272. Example of a Bad Foundation | 328 |
| 273. Construction of the Minimum Value of the Passive Resistance of Earth Pressure | 329 |
| 274. Opposite Hyperbola of Passive Resistance | 329 |
| 275. Counterforts | 331 |
| 276. Fluid Pressure upon Overfall Weir | 331 |
| 277. Relation between the Centre of Gravity of a Trapezium, and a Certain Pole and Polar | 332 |
| 278. Profile of a Reservoir Wall | 332 |

CHAPTER VIII.

THE TUNNEL.

Section I.—Symmetrical Tunnel.

| | |
|---|-----|
| 279. Example | 335 |
| 280. Line of Pressures in a Tunnel Arch | 335 |

Section II.—Unsymmetrical Tunnel.

| Art. | Page |
|--|------|
| 281. Unsymmetrical Tunnels in Sidelong Ground | 336 |
| 282. Necessity of an Unsymmetrical Form in some situations | 336 |
| 283. Tunnel Sole | 337 |

CHAPTER IX.

PROJECTIVE GEOMETRY.

Section I.—Anharmonic Ratio.

| | |
|--|-----|
| I. Principle of Signs in Geometry | 339 |
| II. Lemma. Expression for the Area of a Triangle in Terms of Two Sides and their included Angle | 339 |
| III. Corresponding Points in Two Transversals, cutting the same Pencil of Rays | 340 |
| IV. The Infinitely Distant Point on One Transversal and its Correspondent Point on Another | 340 |
| V. The Constant Rectangle in Two Transversals of the same Pencil of Rays | 340 |
| VI. The Anharmonic Ratio of Four Points in all Transversals of the same Pencil of Rays is Constant | 341 |
| VII. The Anharmonic Ratio of Four Rays passing through the same Four Points of a Straight Line is Constant | 341 |
| VIII. To construct an Anharmonic Ratio of given Value | 342 |
| IX. Definition of Harmonic Ratio, Perspective and Projective | 342 |
| X. Fundamental Anharmonic Ratios | 343 |
| XI. Four Points of a Line determine Twenty-four Anharmonic Ratios | 344 |

Section II.—Homology.

| | |
|--|-----|
| XII. Desargue's Theorem regarding Homologous Triangles | 344 |
| XIII. Another Enunciation of Desargue's Theorem | 346 |
| XIV. The Infinitely Distant Straight Line the Locus of all Infinitely Distant Points which are situated in the Same Plane | 346 |
| XV. Fundamental Theorem of Homology | 347 |
| XVa. Remarks | 348 |
| XVI. Projections of Parallel Rays | 348 |
| XVII. Parallel Rays meet in a Point at Infinity | 348 |
| XVIII. Pencils of Parallel Rays on One Plane meet on the Infinitely Distant Straight in as many Points as there are Separate Pencils | 349 |

| Art. | Page |
|---|------|
| XIX. Plane at Infinity | 349 |
| XX. Definitions of a Complete Quadrilateral and a Complete Quadrangle | 349 |
| XXI. Theorem upon the Projectivity of Complete Quadrilaterals | 349 |
| XXII. Lambert's Problem. Projection of an Irregular Quadri- lateral into a Rectangle | 350 |
| XXIII. In Projecting the Irregular Quadrilateral into a Rectangle, the Locus of the Centre of Projection is the Surface of a Sphere | 351 |
| XXIV. Given Two Rays directed to an Inaccessible Point F , to find another Ray directed to that Point and passing through a given Point | 351 |
| XXVa. The Centre of Homology its own Correspondent . . . | 352 |
| XXVb. The Axis of Homology its own Correspondent . . . | 353 |
| XXVIa. The Centre of Homology O is the Centre of Similitude when the Axis s of Homology is at Infinity | 353 |
| XXVib. The Centre of Homology O may Itself be at Infinity . . | 353 |
| XXVic. Theorem | 353 |
| XXVID. The Parabola | 354 |
| XXVie. The Hyperbola | 354 |

Section III.—Harmonic Ratio.

| | |
|---|-----|
| XXVII. Any One Diagonal of a Complete Quadrilateral is divided Harmonically by the Other Two Diagonals | 355 |
| XXVIII. Centre of Segment dividing a Given Straight in Harmonic Ratio | 358 |
| XXIX. Theorem giving Construction of Harmonic Ratio . . . | 359 |
| XXX. Real and Imaginary Solutions of the Foregoing Problem . | 360 |
| XXXI. Double Point | 360 |

Section IV.—Poles and Polars in Conic Sections.

| | |
|---|-----|
| XXXII. Poncelet's Condition of Projectivity | 361 |
| XXXIII. Carnot's Theorem | 362 |
| XXXIV. Removal of Points to Infinity and other Simplifications in Carnot's Theorem | 362 |
| XXXV. Diameter of a Conic Section | 364 |
| XXXVI. Harmonic Division of the Diameter of a Conic Section . | 364 |
| XXXVII. The Relation between the Square of a Semichord, Real or Ideal, and the Rectangle formed by the Parts into which it divides its Corresponding Diameter | 365 |
| XXXVIII. The Diametral Plane in a Cone | 366 |
| XXXIX. Ideal Chord common to different Conic Sections . . . | 366 |
| XL. Ideal Conic Sections through the Summit of a Cone . . . | 366 |

| Art. | Page |
|---|------|
| XLI. Ideal Circular Conic Section through the Summit of a Cone | 267 |
| XLII. Projection of a Conic Section into a Circle | 367 |
| XLIII. Definitions of Pole and Polar | 369 |
| XLIV. Projection of a Conic Section, with Deductions relating to Poles and Polars | 369 |
| XLV. Problems regarding Poles and Polars | 372 |
| Pole and Polar in a Parabola | 373 |
| <i>Section V.—Pascal's and Brianchon's Theorems.</i> | |
| XLVI. Pascal's Theorem | 374 |
| XLVII. Brianchon's Theorem | 376 |
| XLVIII. Constructions based upon Pascal's and Brianchon's Theorems, including Maclaurin's Theorem | 376 |
| XLIX. Points at Infinity in Pascal's and Brianchon's Theorems | 378 |
| L. One, Two, and Three Sides of Pascal's and Brianchon's Theorems becoming infinitely small | 381 |
| <i>Section VI.—Projectivity of Points and Rays.</i> | |
| LI. Three Points, A, B, C , on a Line u , fig. xxxix., corresponding to Three Points, A', B', C' , on a Line u' , being both given, find the corresponding Point D' on u' to any Point D on u | 384 |
| LII. Solution of the above when u and u' are superimposed | 385 |
| LIII. Three Rays, a, b, c , of one Pencil U corresponding to Three Rays, a', b', c' , of another Pencil U' being both given, find another Ray d' of the Second Pencil U' to a given Ray d of the First Pencil U | 386 |
| LIV. United Points in Projective Rows of Points | 386 |
| LV. One United Point | 386 |
| LVI. Constant Rectangle in superimposed Projective Rows of Points and consequences | 388 |
| <i>Section VII.—Involution.</i> | |
| LVII. Involution of Points in Quadrilateral | 389 |
| LVIII. " " Rays " " | 390 |
| LIX. Conjugate Pairs of Elements in Involution | 390 |
| LX. Involution a Special Case of Superposition | 390 |
| LXI. The Centre O of Involution | 390 |
| LXII. Double or United Points in an Involution | 391 |
| LXIII. Existence of Two Double Points in Involution | 391 |
| LXIV. Problem.—To Construct an Involution of the First Species | 391 |

| Art. | Page |
|---|------|
| LXV. To construct an Involution of the Second Species | 392 |
| LXVI. To find the Central Point O of the Involution of which we have two pairs of Conjugate Points | 393 |
| LXVII. The Revolution of a Right Angle forms an Involution of Second Species | 393 |
| LXVIII. Poncelet's Proof of Art. lvii. | 393 |

Section VIII.—Forms Projective in the Circle.

| | |
|---|-----|
| LXIX. General Theorem | 394 |
| LXX. Property of a Mobile Tangent cutting two Fixed Tangents . | 395 |
| LXXI. A Tangent moving between two Fixed Tangents the Rays generate two Projective Pencils | 396 |
| LXXII. Four Tangents a, b, c, d , of a Circle are cut by a Fifth Tangent in Four Points of which the Anharmonic Ratio is Constant | 396 |
| LXXIII. Case of Involution in a Circle | 397 |

Section IX.—Forms Projective in the Conics.

| | |
|--|-----|
| LXXIV. General Theorem | 397 |
| LXXV. Preceding Section extended to Conics | 398 |
| LXXVI. Article lxix. extended to Conics generally | 398 |
| LXXVII. Article lxxii. extended to Conics | 400 |
| LXXVIII. Extension of lxxvii. to the Parabola | 402 |
| LXXIX. Desargues' Theorem in Conics | 403 |
| LXXX. Reciprocal to Desargues' Theorem in Conics | 403 |
| LXXXI. Application of Desargues' Theorem in Conics when no Points are at Infinity | 404 |
| LXXXII. Poncelet's Demonstration of above Theorems | 404 |

ERRATA.

- Page
9. Fig 5. For " $\Sigma'Px : h = R\tilde{x}$ " read " $\Sigma Px : h = R\tilde{x} : h$."
33. Fig. 16. The hatching to the right extending vertically upward from e' should be continued till it meets t .
44. Line 21 from top, for " $E'D'C$ " read " $E'D'C$."
46. In fig. 27a, within the boundary $ABDC$, for " O " read " O ," and accentuate the small letters denoting angles.
67. Fig. 39a. For " a " read " a'' ."
The line " $\overline{O, 2}$ " of fig. 39a is the " a " of fig. 39d.
89. Fig. 62. For "lowermost left-hand a " read " a' ."
97. Line 9 from top, for " f " read " of ."
146. Line 3 from bottom, after "neutral axis" add "through G ."
171. First line of note,
for " $l \int_0^a (1 - \frac{z}{a}) z^2 dz$ " read " $l \int_0^a (1 - \frac{z}{a}) z^2 dz$."
212. Line 9 from bottom, for "A Geometrical Problem" read "A Geometrical Theorem."
215. Line 13 from top, for " D " read "5."
243. In fig. B(b), for " c " read " e ."
257. Fig. 145. For " $\frac{\Sigma m}{2}$ " read " $\frac{\Sigma m}{2a}$ ".
265. Line 6 from bottom, for "96" read "101."
279. Line 5 from top, for " y_g " read " y_g ".
327. Fig. 181. For " n scale" read " u scale."
329. Line 8 from top, for " $\pm (\phi + \phi')$ " read " $\mp (\phi + \phi')$."
333. Lines 1 and 2 in force polygon consider parallel with forces 1 and 2 in cord polygon.
332. Line 12 from top, for "pole" read "antipolar."
Line 15 from top, for "polar" read "antipolar."
356. Line 8 from bottom, for "left" read "right."
- Plate Va. For "7, 8" read "7, 7" wherever it occurs.

ERRATA.

- PAGE
31. Line 12 from top. *For E read A.*
73. Lines 9 and 12 from bottom. *For c read b.*
89. Line 4 from top. *For joints read joint.*
115. Add to art. 99 as follows:
 "Again, when c and c' are unequal, we have (96, ii.)
 $cA = c'A'$, then

$$pc' = \frac{c'Zy}{\Sigma mz^2} = \frac{c'Zy}{c^2A + c'^2A'} = \frac{c'Zy}{cc'A' + c'^2A'} = \frac{Zy}{cA' + c'A'} = \frac{Zy}{rA'} "$$
132. Line 7 from top. *For Σmx^2 read Σmz^2 .*
133. " 9 " bottom. *For $\Sigma_1^3 x$ read $\Sigma_1^5 x$.*
162. Lines 6 and 7 from top. *For ordinates read ordinate.*
169. Line 7 from bottom. *For $\frac{h}{b}$ read $\frac{h}{6}$.*
201. " 12 " " *For the right-hand expression substitute the less ambiguous form—*

$$= \frac{1}{EI} \sum_0^y M_1 \Delta y \cdot y + M_2 \Delta y (y - \Delta y) + M_3 \Delta y (y - 2\Delta y) + \dots M_n \Delta y \cdot \Delta y.$$
203. Line 8 from top. *For Σ_{II}^{VII} read Σ_{II}^{VI} .*
 " 6 " bottom. *For (e'_i) read e'_i .*
204. " 5 " top. *For $z = c + \frac{1}{h \cdot a \cdot \ddot{e}_i}$ read $z = c + \frac{EI}{h \cdot a \cdot \ddot{e}_i}$.*
205. " 17 " " *For 134 read 143.*
218. Lines 11 and 13 from top. *For 133 and 135 read 143 and 144.*
226. Line 11 from bottom. *For nh read nh' , and for CDH read CD .*
234. " 16 " " *Cancel 6.*
242. Bottom line. *For $U'i$ read $U'i_b$*
254. Line 15 from top. *For that measures read that 34" measures.*
 " 16 " " *For 3' read 3.*
 " 19 " " *For 33" read 33.*
 " 20 " " *For 3"4 read 34"*
257. Fig. 145. *For $\frac{\Sigma mz^2}{2}$ read $\frac{\Sigma mz^2}{2abc}$.*
259. Line 10 from bottom. *Cancel remainder of sentence after "zero."*
279. " 5 " top. *For y_g read y_g^s .*
306. " 15 " " *For OB read OC .*
317. " 6 " " *For indirectly read inversely.*
319. " 2 " " *For 213 read 253.*
393. " 16 " " *For AHA, BHB read AHA', BHB .*

GRAPHICAL DETERMINATION OF FORCES

IN

ENGINEERING STRUCTURES.

CHAPTER I.

PARALLEL FORCES IN ONE PLANE.

Section I.—Introductory.

1.—*Preliminary Problem.*—To find the value, graphically, of any number of pairs of factors: in other words, to find the value of

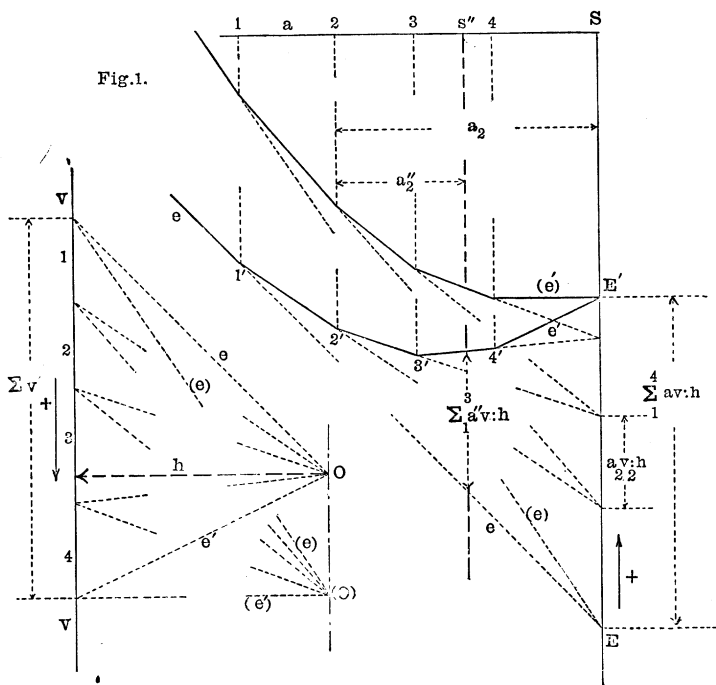
$$a_1v_1 + a_2v_2 + a_3v_3 + \dots a_nv_n,$$

or more concisely, of

$$\Sigma_1^na_v,$$

by means of a force and cord polygon.

Case I. of the Preliminary Problem.—When the factors are all positive.



Upon a horizontal line a , fig. 1, from a point S as origin, lay off the a series of factors, that is, lay off

$$S1 = a_1, S2 = a_2, S3 = a_3 \dots$$

and through the points $S, 1, 2, 3, \dots$ draw vertical lines downwards. In a convenient place draw a line v parallel to these verticals, and upon it lay off the v factors successively, viz.,

$$v_1 = 1, v_2 = 2, v_3 = 3 \dots$$

From a point O at any chosen horizontal distance h from the line v , draw the pencil of rays

$$O[(v1), (1, 2), (2, 3) \dots]$$

to the end points of these factors.

The extreme lines of these rays $O(v, 1)$ and $O(n, v)$ [in figure $O(4, v)$] have been called e and e' ; this pencil of rays and vertical line v will in future be referred to as the force polygon, and the vertical v as the line of weights.

From any convenient point in the vicinity of the line a , draw the ray e parallel to the ray e of the force polygon, intercepting the vertical through the point 1, the end point of a_1 in the point 1', then parallel to the ray $O(1, 2)$ of the force polygon, draw the ray 1'2' intersecting the vertical through 2 in 2': proceed in this manner with all the rays.

Produce 1'2', 2'3', 3'4' till they meet the vertical line s drawn through S , then, by alternate similar triangles, we have, for example,

$$h : v_2 :: a_2 : \frac{a_2 v_2}{h}.$$

This particular instance of the multiplication of the factors a and v , is marked on fig. 1.

Summing up all the separate intercepts thus obtained on the line s we find,

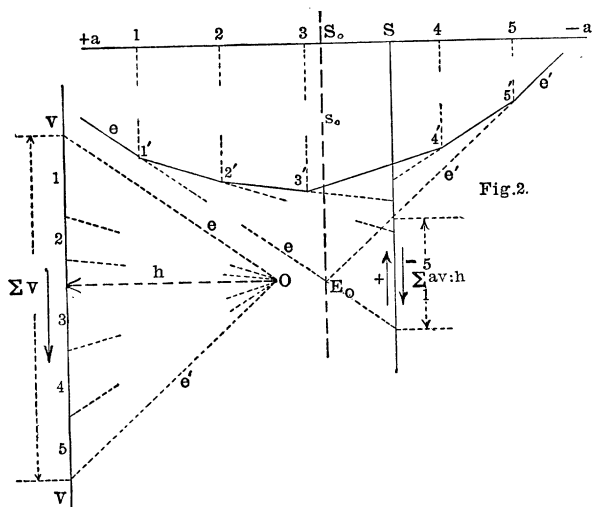
$$E'E = a_1 v_1 : h + a_2 v_2 : h + a_3 v_3 : h + \dots a_n v_n : h = \Sigma_1 a v : h,$$

or in words, the intercept on the line s between the extreme rays e and e' , is equal to the value of the pairs of factors divided by a given constant h .

2. *Definition of Force and Cord Polygon.*—For reasons which will afterwards be given, the figure formed upon the line a shewn in full lines, is called the funicular, cord, or link polygon, and the figure formed upon the line v , as already stated, is called the force polygon, the point O is called its pole, the line h the pole distance, and the line v the line of weights.

3. *Effect of Moving the Pole O in a Vertical Line.*—It is evident that the position of the pole O , while the pole distance h remains the same, does not alter the summation of the intercepts. This has been shewn on fig. 1 by more finely drawn lines, the cord polygon being supposed begun from the point E' , with a parallel to the inferior (e') of the two extreme rays (e) and (e'), so as to ensure coincidence in each value of the two series of intercepts.

4. *Case II. of the Preliminary Problem.*—When one series a of the factors contains both positive and negative quantities—



Lay off on the line a , fig. 2, the positive a factors on the one side (say left) of origin S and the negative on the other side. Complete as in Case I. A little consideration of the figure will make apparent that, as in the first case, the intercept on the line s between the extreme rays e and e' is equal to the value of the pairs of factors divided by a given constant h . We will not therefore give a formal demonstration.

5. *Case III. of Preliminary Problem.*—When there are positive and negative values in both series of factors, fig. 3,

Proceed as in Case II. for the a factors, measure the v factors downwards for positive values, and upwards for negative values, and then completing the force and cord polygons, we find again as in the first and second cases, that the intercept on the line s between the extreme rays e and e' , is equal to

$$\sum_1^n av : h$$

which we leave without formal demonstration.

6. *The Scales of Force and Cord Polygons are not necessarily the same*, that is, h and the v factors may be drawn to one scale, while the a factors are drawn to another scale.

For, let unit on *force* polygon scale measure n units on *cord* polygon scale, then

$$\begin{aligned} & \text{Intercept on } \textit{cord} \text{ scale} \times \frac{\text{pole distance on } \textit{cord} \text{ scale}}{n} \\ &= \frac{\text{Intercept on } \textit{cord} \text{ scale}}{n} \times \text{pole distance on } \textit{cord} \text{ scale} \\ &= \text{Intercept on } \textit{force} \text{ scale} \times \text{pole distance on } \textit{cord} \text{ scale.} \end{aligned}$$

8. *Inversion of Force and Cord Polygon*—In our figures (1, 2, 3) we have placed the pole O of the force polygon to the right of the line of the v factors, but it may be convenient to place it to the left, and it is evident that with the same arrangement of positive and negative values we obtain a cord polygon in an inverted position.

9. *Summation of the Intercepts $av : h$ Independent of the Order of their Seizure.*—In the figures given, the terms of the series

$$a_1v_1, a_2v_2, a_3v_3 \dots a_nv_n$$

in the summation have been arranged in the order of the algebraical values of the a factors; but the summation $\Sigma av : h$ is not affected by a different arrangement.

In fig. 4 we have a series of six pairs of factors

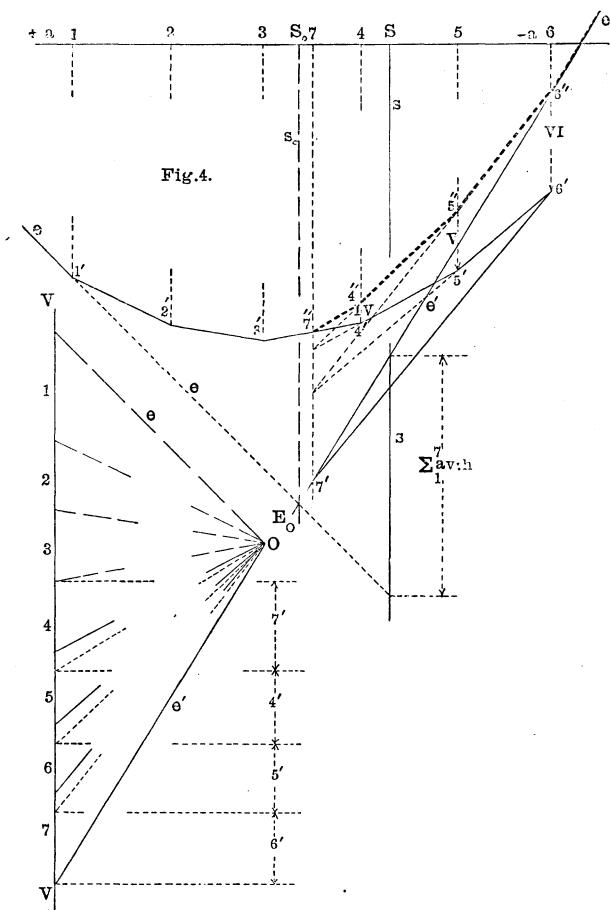
$$a_1v_1 + a_2v_2 + \dots a_6v_6,$$

and it is required to add a_7v_7 to this series, a_7 having a value between a_3 and a_4 .

The resulting intercept giving $\Sigma_1^7 av : h$ is necessarily the same whatever order may be followed, that is

$$\begin{aligned} & a_1v_1 + a_2v_2 + \dots a_6v_6 + a_7v_7 \\ &= a_1v_1 + a_2v_2 + a_3v_3 + a_7v_7 + a_4v_4 + \dots a_6v_6 = \Sigma_1^7 av. \end{aligned}$$

10. *In Two Cord Polygons to the same Pairs of Factors but differing in the Order of Seizure of One of these Pairs, their corresponding Sides intersect in a Straight Line, this Straight Line being the Vertical through the extremity of the Interpolated Factor.*—The theorem of paragraph 9, otherwise a truism, has been introduced to bring out this important property of corresponding cord polygons.



In fig. 4 the full lined cord polygon follows the first arrangement; the second arrangement, in so far as it differs from the first, is shown by strong broken lines, the prolongations of corresponding sides of both cord polygons by fine broken lines.

As the ray e' in the force polygon is common to both arrangements (for Σv is the same in both) the second, or e' extreme rays of both cord polygons must at least be parallel. They must also coincide, for beginning at the 7" of divergence of the two cord polygons; in the triangle 7", 4", 4', and in the figure 4"5"5'4' produced to form a second triangle, both having the

same base $4''4'$, we have two triangles similar to the triangles in the force polygon having respectively for bases $7'$ and 4 with the same vertex O .

But the following reciprocity is evident without formal demonstration. Triangles having the same vertex and bases upon the same straight line are reciprocal to similar triangles upon the same base and whose vertices will lie upon a straight line parallel to the base. This reciprocity likewise holds good when, as in our combined cord polygons (fig. 4), the second triangle being constructed on the same base IV , as the first, it is then extended backwards to a new and parallel base V , upon which the third triangle is drawn, which again is extended backwards to a new and parallel base VI , upon which the fourth is drawn.

These triangles, by the above theorem of reciprocity, have all their vertices on the vertical through 7 , whence "In two cord polygons, &c." This theorem will afterwards be generalized (arts. 50–52).

Section II.—Line of Action of Resultant of Parallel Forces in one Plane, and Centre of Gravity of a Lamina.

11. *Line in which $\Sigma av : h = 0$.*—The following is an important corollary to the preliminary problem. When (figs. 2, 3, 4) the vertical s is removed parallel to itself till it passes through the intersection of the two extreme rays e and e' giving a new origin S_0 and a new vertical s_0 we have then

$$\Sigma av : h = 0 \text{ or } \Sigma av = 0.$$

12. *Line of Action r of the Resultant R of Parallel Forces acting in one Plane.*—The usual equation for the line of action r of the resultant R in works on statics is

$$\bar{x} \cdot \Sigma P = \bar{x} \cdot R = P_1 \cdot x_1 + P_2 \cdot x_2 + P_3 \cdot x_3 + \dots P_n \cdot x_n \dots (1)$$

where \bar{x} is the distance of r from the origin S (fig. 5).

When the origin S has been so chosen that $\bar{x} = 0$ the above expression becomes

$$0 = P_1 x_1 + P_2 x_2 + \dots P_n x_n = \Sigma P x.$$

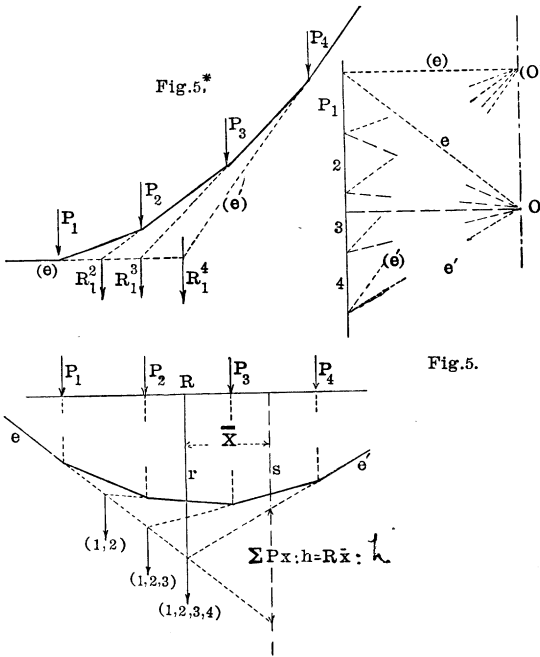


Fig. 5.

Now if for a in the preliminary problem we substitute x and for v we substitute P , then the line r is the line s_0 of paragraph 11, and our problem is reduced to finding the line s_0 of the foregoing preliminary problem.

Fig. 5 is the solution of this problem, for the resultant R of four forces P_1, P_2, P_3, P_4 .

Similarly, the vertical line through the point (1, 2, 3) is in the line of the resultant of the first three forces, and the vertical through the point (1, 2) is in the line of action of the resultant of the first two forces.

Taking any other point S in vertical s as origin from whence to measure the factors x and making $SR = \bar{x}$, the intercept on s of the extreme rays e and e'

$$\Sigma Px : h = \bar{x} \cdot \Sigma P : h = \bar{x} \cdot R : h,$$

because by alternate similar triangles

$$h : \Sigma P :: \bar{x} : \frac{\bar{x} \cdot \Sigma P}{h}$$

the two similar and alternate triangles being (1) in the force polygon, that having ΣP for its base and h for its altitude, and (2) in the cord polygon, that having $\Sigma Px : h$ for its base and \bar{x} for its altitude.

When e coincides with h in the force polygon, removing the pole O to (O) (fig. 5) we obtain a form of the cord polygon fig. 5*, highly useful in the consideration of the suspension bridge and arch.

13. *Centre of Gravity of a Lamina.*—The usual mode of stating the position of the centre of gravity of a thin lamina of uniform density in works on statics is

$$\left. \begin{aligned} \bar{x}\Sigma m &= m_1x_1 + m_2x_2 + \dots + m_nx_n \\ \bar{y}\Sigma m &= m_1y_1 + m_2y_2 + \dots + m_ny_n \end{aligned} \right\} \dots (2)$$

where m is the weight or area, and x_1y_1 the co-ordinates of the centre of gravity of one of its parts, \bar{x}, \bar{y} the co-ordinates of the centre of gravity of the whole.

14. *Centre of Gravity of a Lamina having an Axis of Symmetry, with Example.*—If a lamina has an axis of symmetry, taking this for the axis of x , then $\bar{y}\Sigma m = 0$ and the centre of gravity lies in this axis, whence analytically the problem in this case is reduced to finding \bar{x} , and in our construction to finding the line in which $\bar{x} = 0$.

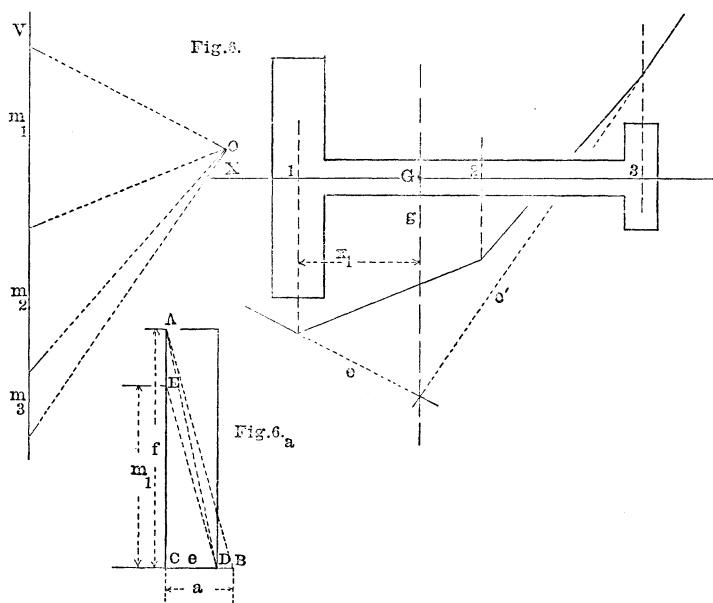
Example.—Let it be required to find the centre of gravity of a lamina as in fig. 6.

In this example the line of x goes through the axis of symmetry laid horizontally, m_1 in the force polygon being made proportional to the area of the lower flange, m_2 proportional to the area of the web, m_3 proportional to the area of the upper flange, x_1, x_2, x_3 being the centres of gravity respectively of these parts. The pole O is arbitrary. From this force polygon construct the cord polygon and the line g drawn vertically through the intersection of the extreme rays cutting the axis of x in G , giving

$$\bar{x} \cdot \Sigma m = \Sigma mx = 0.$$

G is thus the centre of gravity required.

15. *Representation of Areas by Lines, or Reduction of Areas*



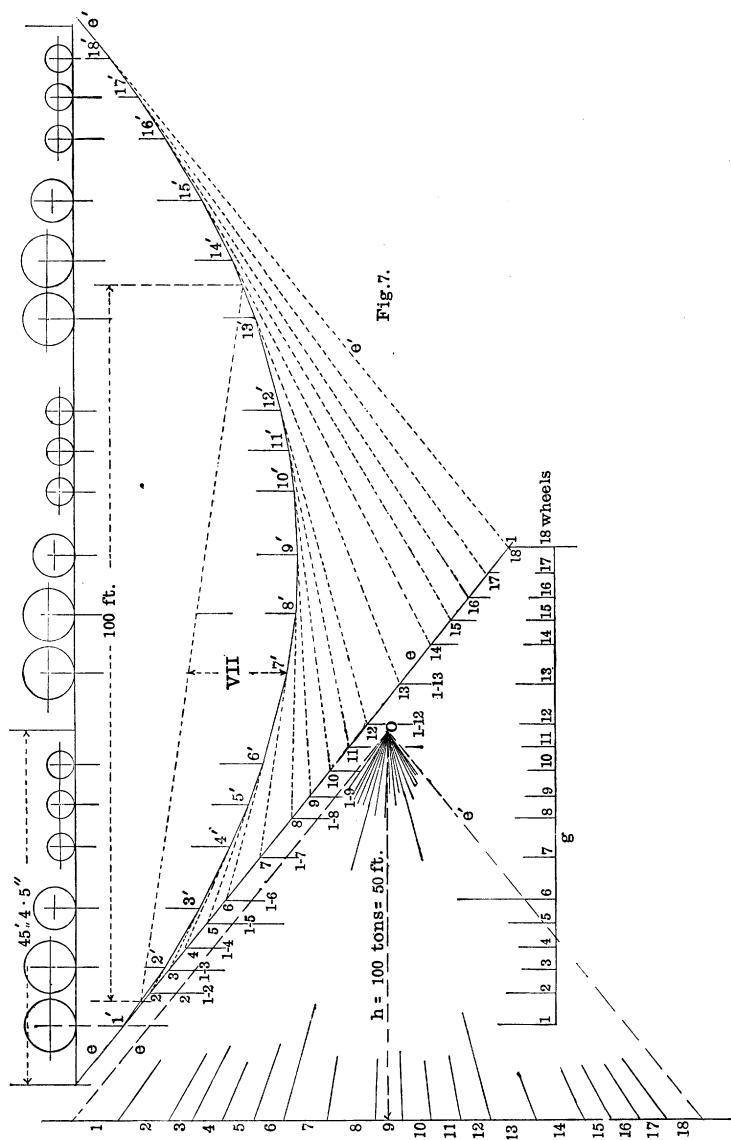
upon a given Base a .—We confront here for the first time the problem of representing a series of areas by lines proportional to those areas. In order to obtain these lines the areas must be transformed into rectangles, all having the same breadth a of base, unity being in general too small to be convenient. For instance, fig. 6, in order to obtain the lengths m_1, m_2, m_3 of the force polygon, a base a (fig. 6a) equal to twice the breadth of the web, was chosen, and the transformation of the lower flange into the line m_1 so that $m_1 \times a = \text{area of lower flange}$, is an example of its employment. $ACD = ef$ is the rectangular area of the flange whose height, $AC = f$, breadth $CD = e$. Lay off $CB = a$, draw DE parallel to BA . The rectangle $BCE = m_1 a = \text{rectangle } ACD = ef$ for

$$a : e :: f : \frac{ef}{a} \\ : m_1,$$

whence

$$m_1 \cdot a = ef.$$

Reducing thus as many areas 1, 2, 3 . . . as are in the series to the base a , they can be represented by their reduced heights $m_1, m_2, m_3 \dots$



It must be remembered that this process is of constant occurrence in the sequel.

16. *Finding the Verticals through the Centres of Gravity of any successive Number of Wheels of Locomotives and Tenders (up to Eighteen, in our figure) as they might be upon a Bridge on its proof trial.*—This is fully worked out on fig. 7. The vertical line through the centre of gravity of the eighteen wheels is the vertical drawn through the intersection (18, 1) of the extreme rays. The vertical line through the centre of gravity of, say, the first thirteen wheels is the vertical through the point of intersection 13 of the lines e and $13'$, $14'$. In the same manner the vertical through the centre of gravity of any consecutive number of wheels may be obtained.

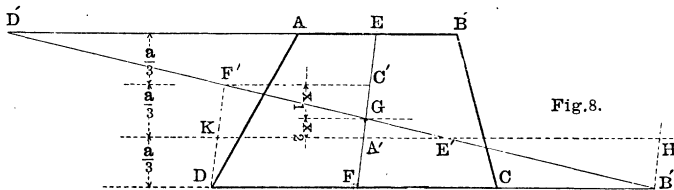


Fig. 8.

17. *Centre of Gravity of a Trapezium.*— $ABCD$, fig. 8, is a given trapezium of height a . The full lines in the figure are those required by the construction, the broken lines are necessary to demonstration.

Construction.—Bisect the two parallel sides AB and CD in E and F respectively. Join EF , prolong AB on one side until the part produced is equal to CD , and prolong CD on the other side until the part produced is equal to AB . Join the free extremities D' , B' of these produced lines by a straight $D'B'$; the point G where this line cuts EF is the centre of gravity of the trapezium.

Demonstration.—Suppose the diagonal DB drawn, then the area of the two triangles ABD , BDC are respectively equal to $a \cdot AE$ and $a \cdot FC$, and the centres of gravity are at the distance $\frac{a}{3}$ from their respective bases; draw $F'C'$ and $A'E'$ parallel

to these bases and at distances from them equal to $\frac{a}{3}$, making each of them equal to half the base of the alternate triangle; then join $F'E'$, $E'F'$ cuts EF in the required point G .

For, by elementary principles of statics, the centre of gravity of the trapezium lies in EF , and if we suppose it so placed that AB and CD are vertical, the weight of the triangular lamina ABD acts downwards in $C'F'$ and that of BCD in $E'A'$ and $C'A'$ is cut inversely in G , so that a vertical through G would give

$$EB \cdot x_1 = FC \cdot x_2, \text{ or } EB x_1 + FC x_2 = 0,$$

whence G is the point required.

It is easy to prove that G is the point given by the first construction.¹

18. *The Centre of Gravity of an Irregular Quadrilateral Lamina, ABCD.*

Construction.—Divide the quadrilateral by the diagonal AC into two triangles. Find the centre of gravity G_1 , of the triangle ADC , bisecting for this purpose the diagonal AC in E and the side DC in K . Through G_1 draw G_1G_2 parallel to DB . Exchange the segments DF , BF by making $BH = DF$. Join HE . The point G where HE cuts G_1G_2 is the centre of gravity required.

Demonstration.—Join BE completing the triangle DEB and by similar triangles we have the proportion

$$DE : G_1E :: BE : G_2E$$

¹ Complete the figure as shown, then DF' is parallel to EF because by construction $C'F' = DF$, and drawing $B'H$ parallel to EF we have

$$KF' = KD = B'H$$

whence by similar triangles

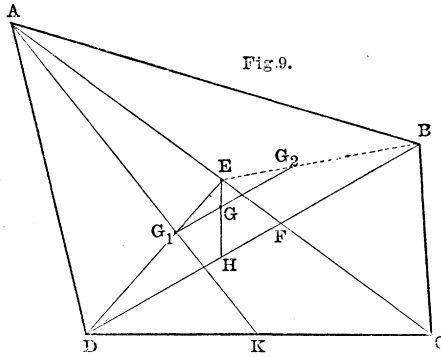
$$EH = EK = E'A' + C'F' = BE + CF.$$

Adding $E'A'$ or BE to both sides,

$$A'E' + E'H = FB' = 2 \cdot BE + CF$$

taking FC from both sides $CB' = AB$.

In the same manner might be proved $AD' = CD$.



and as by construction

$$G_1E = \frac{1}{3} DE$$

we have

$$G_2E = \frac{1}{3} BE$$

G_2 is therefore the centre of gravity of the triangle ABC . We have also the proportion

$$DH : G_1G :: BH : G_2G$$

also

$$DH : \triangle ABC :: BH : \triangle ADC$$

whence

$$\triangle ABC : G_1G :: \triangle ADC : G_2G$$

or the line joining the centres of gravity of the two triangles of which the quadrilateral is composed has been divided inversely as their areas in the point G , whence, by statics, G is their common centre of gravity.

19. *Centre of Gravity of a Lamina having no Axis of Symmetry, with Example.*—In finding the centre of gravity of a lamina having an axis of symmetry, it is only necessary, as we have seen, to find the line in which, taken as through the origin $\bar{x} = 0$, but in order to find the centre of gravity of a lamina having no axis of symmetry, it is necessary to find the co-ordinate axes in which $\bar{x} = 0$ and $\bar{y} = 0$, and the point of intersection of the axes, *i.e.* the origin is the centre of gravity required. (14.)

This can evidently be done by taking two directions for the verticals through the centres of gravity of the various parts of the lamina, and having the force polygon twice constructed so that its line of weights v will first be parallel to the one set of verticals and then to the other, then the two verticals drawn through the intersections of the two pair of extreme rays, intersect in the required point G .

The construction of the force polygon for the second time may, however, be dispensed with by supposing the lamina turned round through say a right angle.

Example, fig. 10. Let it be required to find the Centre of Gravity of an Unsymmetrical Angle Iron Lamina.—Finding the centres of gravity 1 . 2 . 3 6 of the various trapeziums and triangle of which the lamina is composed (17) reducing these parts to a base a (this is not given) (15), we can form the force polygon $O(1 . 2 . 3 6)$ on the line v parallel to the verticals 1 1', 2 2', 3 3', drawing in the corresponding cord polygon $A, 1' 2' 3' 6'B$, and through the intersection E of its extreme rays drawing the line g parallel to v . Then with a set square upon the pencil of rays of the force polygon and the parallel ruler on the other side of the set square, form a second cord polygon $A' 1'', 2'', 3'' 6''B'$, and then through the intersection E' of its extreme rays draw in the line g' parallel to the new set of verticals. The lines g and g' cut in G , which is the centre of gravity of the lamina.

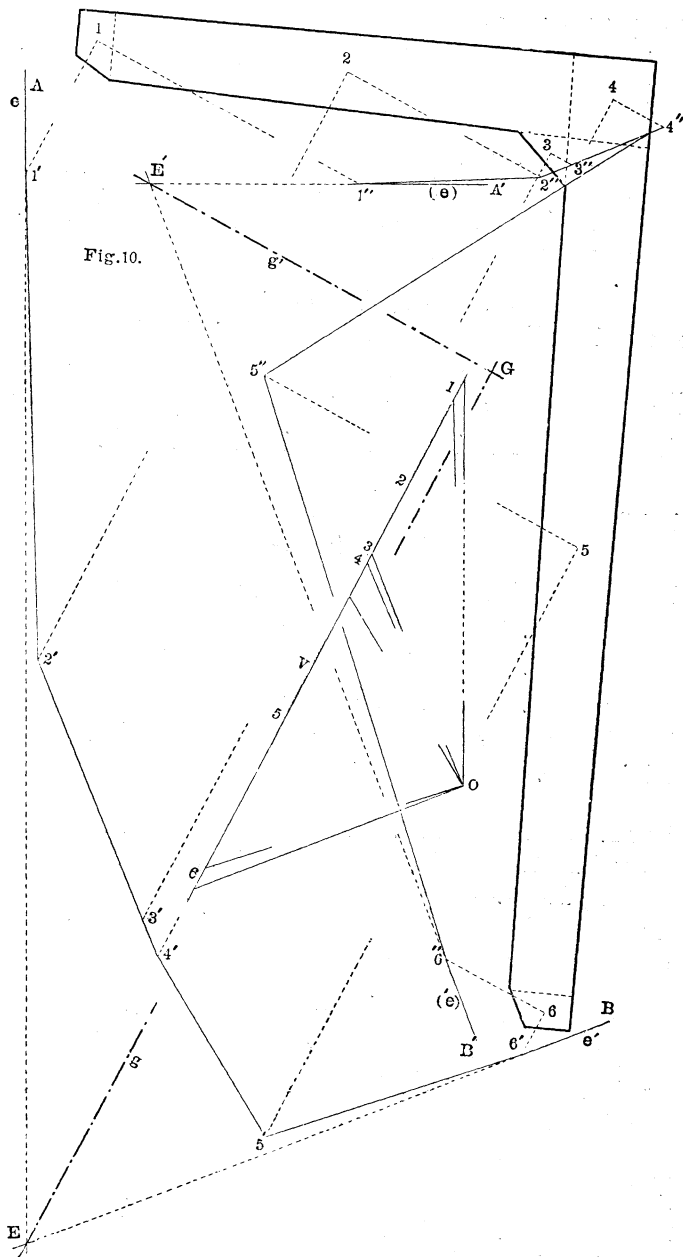
The lines employed in finding centres of gravity may be recognised in several figures further on.

Section III.—Equilibrium in an Ideal Beam between Parallel Forces.

20. *Equilibrium in an Ideal Beam between Three Parallel Forces,* $\begin{matrix} \text{two} \\ \text{one} \end{matrix}$ impressed $\begin{matrix} \text{one} \\ \text{two} \end{matrix}$ reacting.

The conditions necessary to equilibrium among parallel forces are, according to statics

$$\left. \begin{array}{l} \Sigma P = 0 \\ \text{and } \Sigma Px = 0 \end{array} \right\} \dots (3).$$



Let it likewise be observed that in a force and cord polygon in which these two conditions are fulfilled, the number of rays in each is equal to the number of the forces.

In the case of the equilibrium of three parallel forces, the first condition, $\Sigma P = 0$.

The line of weights in the force polygon of fig. 11, is an example of this condition + P_B and + P_A are upwards and - R is downwards, so that

$$\left. \begin{array}{l} \Sigma P = P_B + P_A + R = 0 \\ \text{the second condition is, } \Sigma Px = 0 \end{array} \right\} \dots (4).$$

This condition is shown fulfilled in fig. 11, for $R = P_A + P_B$ goes through the intersection of the extreme rays e and e' for taking the line of action of R for axis through origin

$$Rx = 0$$

and (12)

$$P_A x + P_B x = 0.$$

We have in all cases, known and unknown,

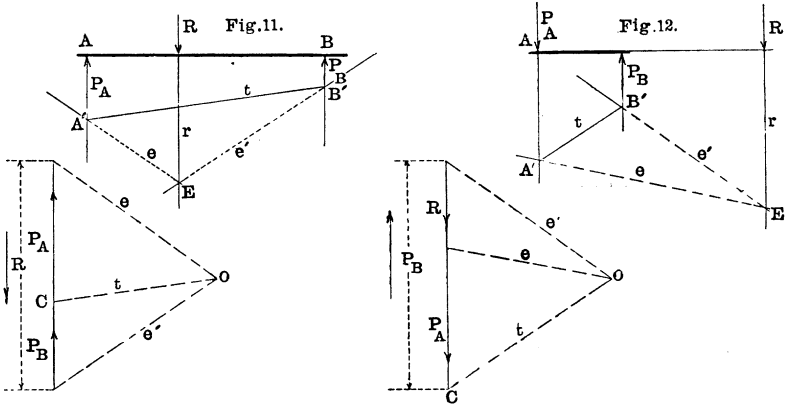
1. three parallel forces,
2. three points of action on the beam,
3. three rays of each polygon,

and of the three forces and three points any four determine the remaining two.

Case I. Figs. 11 and 12. The Three Points A, B, and R of a Beam and the Impressed Force R being given, to determine the Two Supporting Forces.—We have in this case the three rays of the cord polygon given, from which to determine the three rays of the force polygon, when we can immediately determine the two supporting or reacting forces.

For, take any point E on the line of action of R , and from E draw arbitrarily two rays e and e' cutting the verticals through the points of support in A' and B' . Join $A'B'$ by the line t , then e , t , and e' are the three rays of the cord polygon to the impressed force R and the supporting forces P_A and P_B .

For, from the extremities of the given force R in the force polygon, draw the lines c and c' parallel to e and e' of the cord polygon intersecting in O , from O draw t parallel to t in the cord



polygon, t divides R into two portions P_A and P_B which measure the supporting forces, as e , t and e' in this force polygon would give the rays e , t and e' of this cord polygon, and P_A , P_B and R fulfil the two required conditions of equilibrium.

Case II. Figs. 11 and 12. Two Impressed Forces P_A and P_B , and their Two Points of Action A and B upon a Beam being given, to determine the One Reacting Force and its Point of Action.—We have in this case the three rays of the force polygon from which to determine the three rays of the cord polygon, when we can immediately determine the reacting force and its point of action.

For, take any point O arbitrarily for the pole of the force polygon, and draw from it the three rays e , t , and e' , and then, parallel to them, the three corresponding rays of the cord polygon, in which the intersection of the extreme rays e and e' give the point E in the line of action of R (11), and $R = P_A + P_B$, whence the two conditions of equilibrium are fulfilled.

There is no difficulty in solving the remaining cases in which any other four of the three forces and three points are given.

21. Equilibrium in an Ideal Beam between any number of Forces Parallel among themselves.—Having given a series of parallel impressed forces, fig. 13, P_1 , P_2 , P_3 P_n upon a

beam with their points of action and the points of action A and B of the unknown reacting forces P_A and P_B , i.e. the points of support A and B , to determine the supporting forces P_A and P_B .

Having constructed the force and cord polygons for the forces $P_1, P_2, P_3 \dots P_n$ by choosing any point O for pole of the force polygon, produce the extreme rays e and e' of the cord polygon to their intersection in E , and join the points of intersection by e and e' of the verticals through the points of support by the line t , and draw the corresponding ray t in the force polygon parallel thereto. t divides $\Sigma_1^n P$ into two parts P_A and P_B , the supporting forces required.

For (20 and 12) they are the supporting forces of the resultant R acting in the line through the intersection E of the extreme rays e and e' .

t will be called the closing line of the cord polygon. P_A and P_B are the closing lines of the force polygon. This is the simplest form of forces in equilibrium forming an inclosed polygon.

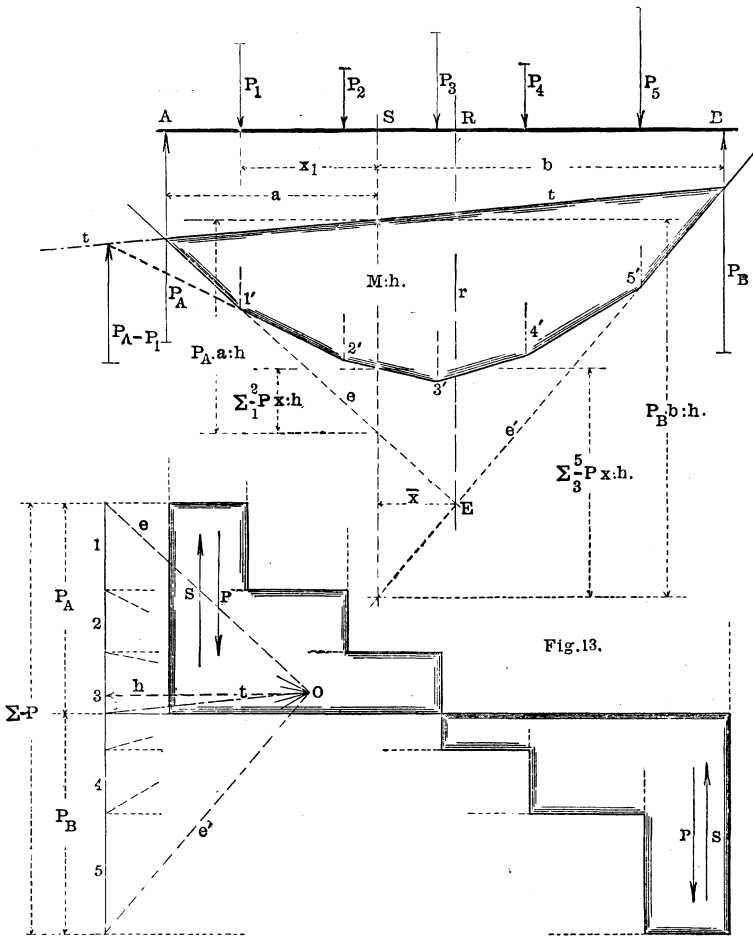
Section IV.—Equilibrium in the Cord Polygon formed by Parallel Forces. ¶

22. *Equilibrium of the Various Members of the Cord Polygon of Parallel Forces.*—If the forces $P_1, P_2, P_3 \dots$ be supposed to be weights hung from the points $1', 2', 3' \dots$ of the cord or link polygon it will be seen that every two sides with the weight at their point are in equilibrium, and that therefore the whole is in equilibrium.

In the force polygon, fig. 13, the weight P_1 would by the theorem of the triangle of forces be kept in equilibrium by two forces in directions parallel to the lines e and $1'2'$, forming with P_1 itself a triangle.

This condition is fulfilled by the three lines of the force polygon, viz., P_1 , on the line of weights, the rays e and $O(1, 2)$.

Next, suppose the weight 2 hung from the joint $2'$ of the cord polygon, then, for the same reason, it is equilibrated by the tension in the cord $1'2'$ and $2'3'$. This condition is fulfilled by



the three lines of the force polygon P_2 on the line of weights, the rays $O(1, 2)$ and $O(2, 3)$. In like manner the tension in the cord $3'4'$ is measured by $O(3, 4)$, and so forth till we arrive at the cord e' . There again P_B , e' and t are in equilibrium, and their values form a triangle in the force polygon. In like manner t , P_A and e are in equilibrium, whence, each part being in equilibrium, the whole is in equilibrium.

Our cord polygon is therefore in complete equilibrium and is the well known funicular polygon.

23. *Alteration of Sign in Tensions of Links of Cord Polygon by Inversion of its Position.*—Considering the reactions P_A and P_B as positive, and the downward forces 1, 2, 3 . . . as negative, the cord is in a state of tension and is concave towards the direction of the action of the forces; but had we placed the pole O to the left of v , we should have obtained a cord or link polygon convex toward the action of the forces, and the links would have been in a state of compression.

As e , t and e' form the cord polygon of the supporting forces (20), t is in the first case in compression, for the cord polygon is convex to its forces P_A and P_B , and in the second case in tension, for the cord polygon is concave to the forces P_A and P_B .

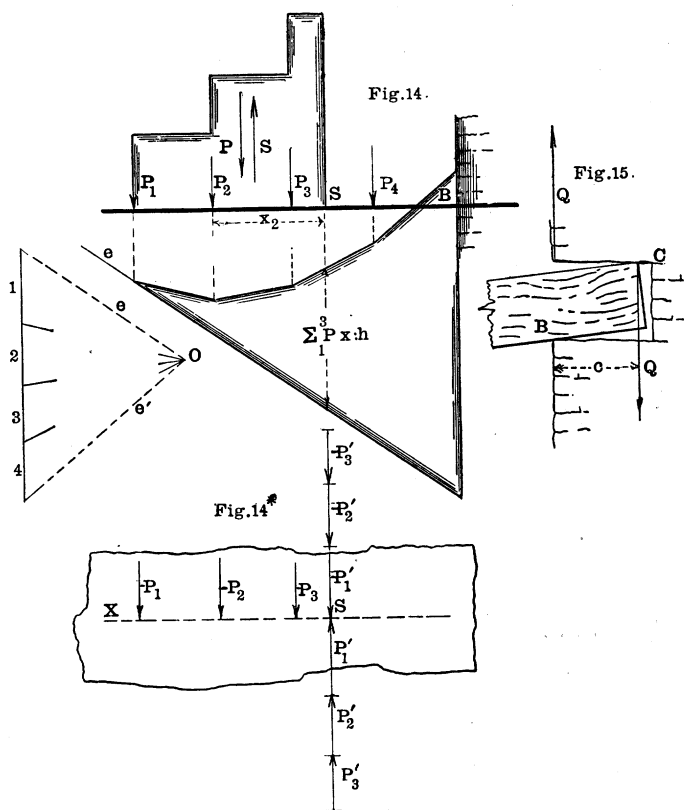
24. *The Horizontal Components of the Tensions in all the Links of a Cord Polygon have the same Value.*—For resolving the tension of any link into a vertical and horizontal component, we perceive that its horizontal component is in all cases h , viz., the pole distance.

Section V.—Action of Parallel Forces in one Plane in a Beam.

25. *Parallel Forces in One Plane in a Beam, act upon it by Shearing Force and Bending Moment.*—We will take first, not necessarily, but for simplicity's sake, impressed forces all of one sign.

In figs. 14 and 14* dealing, first with P_1 only, we require its action upon the same beam or rigid body at any other point S .

If, at S we place two forces $+P_1'$ and $-P_1'$ parallel to P_1 and equal to it in value but of opposite signs, the equilibrium of the body is not disturbed thereby. We have then a force $-P_1'$ acting at S , and a couple composed of the two equal and opposite forces $-P_1$ and $+P_1'$ with arm x_1 , (reckoning S the origin and SX perpendicular to the action of the forces) whose moment is therefore $-P_1x_1$. If again at S in the same manner we place two forces $+P_2'$ and $-P_2'$ each equal and parallel to $-P_2$ we obtain another couple $-P_2x_2$ and thus we proceed to any number of forces.



Note that the sign of the couple is negative, as the turning action is in the direction contrary to that of the hands of a watch.

The action at S of the impressed forces is of two kinds:—

1. There is a direct force consisting of the sum of the applied forces, viz., $-(P_1' + P_2' + P_3' + \dots)$, which the beam or body being fixed at S is resisted by the equal and contrary reaction $+(P_1' + P_2' + P_3' + \dots)$. This is called shearing force.

2. There is a couple consisting of the sum of the couples formed by the individuals, viz.:—

$$-\sum_1^n Px = -(P_1x_1 + P_2x_2 + P_3x_3 + \dots).$$

This is called bending moment.

26. *Reasons for the Names Shearing Force and Bending Moment to the Action of Forces on a Beam.*—The first of these, or the direct force upon any point of a beam, is called the shearing force at that point, because in combination with its reaction it tends to shear the beam; with it is evident an intensity of P , at all points from x_1 to x_2 , of $P_1 + P_2$ at all points from x_2 to x_3 of $P_1 + P_2 + P_3$ at all points from x_3 to S (fig. 14).

This intensity is shown by the upper shaded area, fig. 14, and the relative position of action and reaction by $-P\downarrow\uparrow S$.

The second of these, or the couple ΣPx , is called the bending moment, because it tends to bend the beam around S .

27. *Measure of Shearing Force and Bending Moment generally by Means of Force and Cord Polygons.*—Referring to fig. 14, the measure of the shearing force can be obtained directly from the line of weights v of the force polygon, so simply as to require no explanation.

The measure of the bending moment is obtained by the formation of a force and a cord polygon having the forces P_1, P_2, P_3, \dots arranged on the line v and any value h being fixed on for the pole distance; a corresponding cord polygon is then constructed and (1) the intercept between the extreme rays (extreme in reference to any given point S) gives $\Sigma_1 Px : h$, reckoning S as origin.

28. *Shearing Force and Bending Moment in a Beam fixed at One End.*—We have, in the two foregoing paragraphs, for the sake of simplicity of figure, taken impressed forces of only one sign into consideration, but it is evident that the conclusions therein given are applicable to a combination of forces of either sign, for (4 and 5) the same conclusion is still true, viz., that the intercept between the extreme rays (extreme with reference to the point S selected) gives $\Sigma Px : h$, and it is also evident that ΣP represents the shearing force, taking the signs of the individual P into account.

We conclude, therefore, that in a beam fixed at one end (and therefore fixed by its rigidity at every point of its length) and impressed by parallel forces, the intercept of the ordinate taken perpendicular to the beam at any point S of the beam, between

what are for that point the extreme rays of the cord polygon, measures the bending moment : h also.

The shearing force at any point S of a beam is the sum of the forces having regard to sign between its free end and the point S .

For the scale with which to measure bending moment : h and to measure h , see (7).

29. *The Compound Nature of the Measure of Bending Moment.*

—The nature of this bending moment is denominated by a compound word derived from the units of weight and length employed ; thus, if the weights are given in tons and the lengths in feet or in inches, we use the words foot tons in the one case and inch tons in the other ; thus 5 tons with a leverage $x = 4$ feet from the selected point S gives a bending moment of 20 foot tons or 240 inch tons.

30. *Moment of Encastrement or Inbuilding of a Beam.*—When the end of a beam is fastened in a wall it is said to be encastre or inbuilt. The moment of inbuilding is equal to ΣPx , B being origin (fig. 14).

It is produced, in general, by the support given by the points B and C (fig. 15), from the prolongation of the beam in the wall. For equilibrium we must have

$$\Sigma Px = Qc,$$

c being approximately the length of the beam engaged in the wall.

The point B is also subjected to the vertical reaction of $-\Sigma P$ which call ΣP . The forces $Q + \Sigma P$ being developed by the masonry of the wall, it is always necessary to verify that these forces do not exceed the resisting capacity of the wall, nor compromise its stability.

31. *Shearing Force and Bending Moment of a Beam supported at Both Ends.*—Let AB (fig. 13) be a beam supported at A and B sustaining the downward pressures P_1 to P_5 (downward pressures being given only for simplicity of figure, as the demonstration does not exclude upward pressure), and the two

upward supporting pressures P_A and P_B , then all the exterior forces are

$$P_A + P_B - P_1 - P_2 - P_3 - P_4 - P_5 = 0.$$

Let us now consider the action of these forces at any point S of the beam.

Regarding S as origin, and considering the beam as held fast at S (for the beam being in equilibrium it is held fast at all points), and to fix the ideas, let us first consider the left side AS of the beam as under the impressed forces $+P_A - P_1 - P_2$, and the right side as under reacting forces in equilibrium with those impressed just as if it were inbuilt, whence, transferring to S the forces to the left of that point we have:—

First, the direct force $P_A - (P_1 + P_2)$ which is the direct force at that point and with its reaction constitutes the shearing force there. It is measured on the line v of the force polygon by the distance between the points C (omitted on fig. 13, it is the extremity of the line t) and $(2, 3)$, and second, the sum M of the couples, which is

$$M = \Sigma Px = P_A a - (P_1 x_1 + P_2 x_2).$$

The bending moment : h or M : h is measured as formerly:—

First, $P_A \cdot a : h$ is measured by the intercept on the vertical through S between the extreme ray e and the ray t of the cord polygon, which is by Art. 1, or by the proportion

$$h : P_A :: a : \frac{P_A \cdot a}{h}$$

$$\text{equal to} \quad \frac{P_A \cdot a}{h}.$$

Second, $-(P_1 x_1 + P_2 x_2) : h$ measured likewise on the vertical through S between the extreme ray e and the side $2' 3'$ of the cord polygon, the algebraical summation of these two intercepts being

$$P_A \cdot a : h - (P_1 x_1 + P_2 x_2) : h = \Sigma Px : h = M : h,$$

Or, more generally thus stated $+P_A, -P_1, -P_2$, are three v factors of different sign (4) where t is the first extreme ray and

2'3' the second extreme ray with regard to S , giving an intercept on the vertical through S , $= M : h$ (marked on fig. 13).

In the same manner consider now the right side SB of the beam as under impressed forces, and the left side AS as under equilibrating reacting forces, then at S

$$M : h = \Sigma Px : h = P_B . b : h - (P_3x_3 + P_4x_4 + P_5x_5) : h.$$

Either of these two expressions gives the value of the couple forming the bending moment at S .

32. *Scale with which to ascertain Bending Moment, and Example.*—In obtaining the bending moment by means of this intercept, and the equation $M = M : h \times h$ or in words

bending moment = intercept \times pole distance

the rules regarding scales (7) must be attended to, viz. :

Intercept on foot scale \times pole distance on ton scale,
or Intercept on ton scale \times pole distance on foot scale.

Example.—Let it be required to find the bending moment and shearing force at some point of a girder of 100 feet span loaded with a series of locomotives and tenders, weight of girder being neglected.

Having, upon the cord polygon of fig. 7, laid off 100 feet horizontally, and having from the extremities of this line let fall verticals meeting the cord polygon in two points, join these two points by a line t . This is the closing line of the cord polygon, and the rays 1' 2' and 14' 13' are its extreme rays, and the bending moment under any wheel, 7 for example, is the intercept VII. multiplied by the pole distance.

If the intercept VII. is measured on the foot scale the pole distance h is measured on the scale of tons, in this instance $h = 100$ tons.

The corresponding extreme rays on the force polygon are $O(1, 2)$ and $O(13, 14)$ and drawing t through O we obtain the two supporting forces.

33. *Preparation of a Diagram of Bending Moments.*—From the above example we see that, if such a diagram as fig. 7 were once carefully prepared, the bending moments for a given loading for an extensive series of spans might be read off when required.

Section VI.—Loci of Maximum Bending Moment in a Beam under Parallel Forces and between Two Points of Support.

34. *Locus of Greatest Bending Moment for any Given System of Weights, in One Given Position on a Beam.*—The point in a beam where the bending moment is greatest for any given system of weights placed upon it is that point where the shearing force changes sign. This occurs where the weights between it and the point of support become greater than the reaction. In fig. 13, the greatest bending moment occurs in the vertical in which P_3 acts. The proposition admits, by means of our construction, of an easy geometrical demonstration.

It is evident that a line drawn parallel to the closing line t through the point in the cord polygon, which is in the vertical of the maximum moment point, cannot again cut the cord polygon, and a moment's perception shows that that point is the vertex of the angle of the two sides in the cord polygon, corresponding to the two rays on either side of the line t in the force polygon, from whence the truth of the proposition readily follows.

The following is an analytical demonstration. In fig. 13, the greatest bending moment occurs in the vertical in which P_3 acts, and the conditions are

$$x_3 = 0$$

$$a = AP_3, \text{ (not the } a \text{ represented in the figure)}$$

$$P_A > P_1 + P_2$$

$$P_A < P_1 + P_2 + P_3$$

writing M for bending moment

$$\begin{aligned} M &= P_A \cdot a - (P_1x_1 + P_2x_2 + P_3x_3) = P_B \cdot b - (P_4x_4 + P_5x_5) \\ &= P_A \cdot a - (P_1x_1 + P_2x_2) = \text{maximum.} \end{aligned}$$

Demonstration.—I. Remove the origin a small distance δ to the left of P_3 , then

$$\begin{aligned} M &= P_A(a - \delta) - (P_1x_1 + P_2x_2) + (P_1\delta + P_2\delta) \\ &= P_A(a - \delta) + (P_1 + P_2)\delta - P_1x_1 + P_2x_2, \end{aligned}$$

and as $P_1 + P_2$ is less than P_A , the value of the expression is diminished.

II. Now remove the origin a small distance δ to the right of P_3 , then

$$M = P_A(a + \delta) - (P_1x_1 + P_2x_2 + P_3\delta + P_1\delta + P_2\delta),$$

and as $P_1 + P_2 + P_3$ is greater than P_A , the expression is again diminished, whence &c.

35. *Positions of Maximum of Greatest Bending Moment, arising from a System of Weights.*—There are two positions upon a beam for a given system of weights at invariable distances apart where the greatest bending moment is a maximum.

I. When the centre of gravity of the system of weights lies over the centre point of the beam.

II. When the weight nearest the centre of gravity is put over the centre point of the beam.

Demonstration.—Let fig. 13 represent the *first* of these two positions, *i.e.* let it represent $P_A = P_B$ and consequently $\overline{RA} = \overline{RB}$, and let the position of P_3 be the origin, and remembering that

$$\frac{P_A}{P_B} = \frac{RB}{RA} = \frac{n \cdot P_A}{n \cdot P_B} \text{ and } \overline{RP}_3 = x_3$$

$$P_A + P_B = \Sigma P$$

$$AP_3 = a.$$

The equation for a maximum

$$M = P_A \cdot a - (P_1x_1 + P_2x_2) \text{ becomes}$$

$$M = P_A \cdot (RA - x_3) - (P_1x_1 + P_2x_2)$$

$$= P_A \cdot nP_B - P_Ax_3 - (P_1x_1 + P_2x_2),$$

an equation which has its maximum value when

$$P_A = P_B,$$

for then it is

$$n \cdot P_A^2 - P_A x_3 - (P_1 x_1 + P_2 x_2).$$

Now for the *second* position. Let the weights be rolled over till P_3 stands over R , that is, a distance to the right of x_3 , the above equation for a maximum becomes

$$\begin{aligned} M &= \left(P_A - \frac{x_3}{n}\right)RA - (P_1 x_1 + P_2 x_2) \\ &= P_A \cdot nP_B - \frac{x_3}{n} \cdot nP_B - (P_1 x_1 + P_2 x_2), \end{aligned}$$

an equation which has its maximum when $P_A = P_B$, for then it is

$$n \cdot P_A^2 - P_A x_3 - (P_1 x_1 + P_2 x_2),$$

remembering that P_A in this last equation, although it is the reaction at A in the first case, is not now that reaction, but that $P_A - \frac{x_3}{n}$ is that reaction, and it is equivalent to the former equation giving the maximum value of the greatest bending moment.

Section VII.—Further consideration of the Two Supporting Forces of a Beam.

36. *Ordinary Method of Obtaining the Reactions of the Two Points of Support of a Beam or Cord under a System of Weights.*—We will simply enunciate this proposition.

Let (fig. 13) the reaction at A be required, then let B , the position of the other support, be the origin, *i.e.* the point from which the values of x are measured; then as the reaction of any weight at any one of the two supports between which it lies is inversely as its distance from that support, then AB being = l

$$\begin{aligned} P_A &= P_n \frac{x_n}{l} + \dots P_4 \frac{x_4}{l} + P_3 \frac{x_3}{l} + \dots P_1 \frac{x_1}{l} = \sum_1^n P \frac{x}{l} \\ &= \frac{1}{l} \cdot \sum_1^n Px. \end{aligned}$$

Again, let the reaction at B be required, then let A be the origin, and

$$P_B = \frac{1}{l} \cdot \sum_1^n Px,$$

x being measured from B in finding the reaction A , and measured from A in finding the reaction B .

37. *Identity of the Ordinary and Graphical Methods of Finding the Reactions in a Beam Loaded between Two Supports.*—The above method is easily identified with the previously unfolded graphical method, for, extending the ray e (fig. 13) to the vertical through B ; we have, reckoning B the origin, the intercept between t and e on this vertical = $\sum_B^A Px : h$ (1), and at a horizontal distance h from E draw a vertical intercepted by t and e . We obtain the value of this intercept by means of the following proportion

$$l : \frac{\sum Px}{h} :: h : \frac{\sum Px}{l},$$

the triangle formed thus by t , e , and having a height h being equal and alternately similar to the triangle formed by the corresponding lines t , e , and h in the force polygon.

Section VIII.—Additional Forces brought upon a Beam and Forces taken off a Beam considered.

38. *The Additional Bending Moment brought upon any Point of a Beam by the Interpolation of an Additional Weight at that or any other Point of the Beam.*—If the cord polygon of the additional weight has been separately constructed to the same pole distance h as the cord polygon of the previous weights, the sum of the two ordinates, at any point, will measure the total bending moment : h at that point. For convenience, the two cord polygons can be easily applied to each other as in (fig. 15 bis), which requires no formal elucidation.

39. *Forming a Cord Polygon with Additional Interpolated Forces.*

I. *One Interpolated Force, as in fig. 4.*—In this case let s be

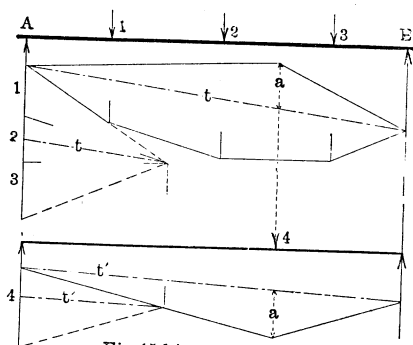


Fig. 15-bis.

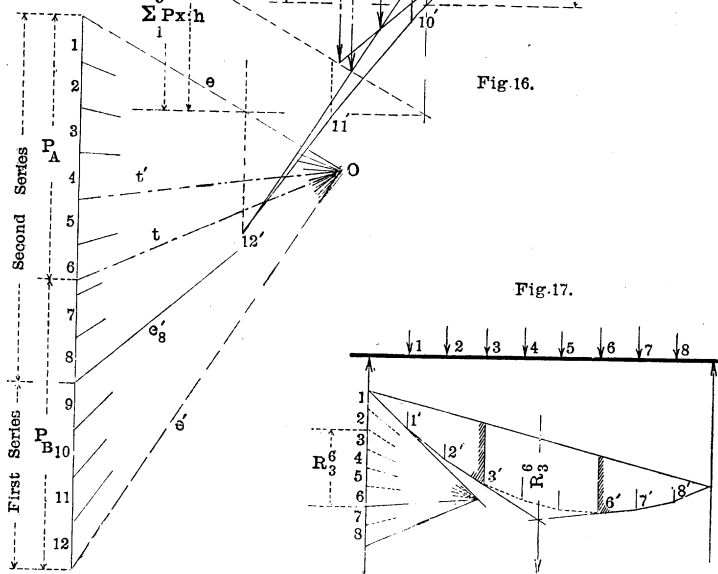
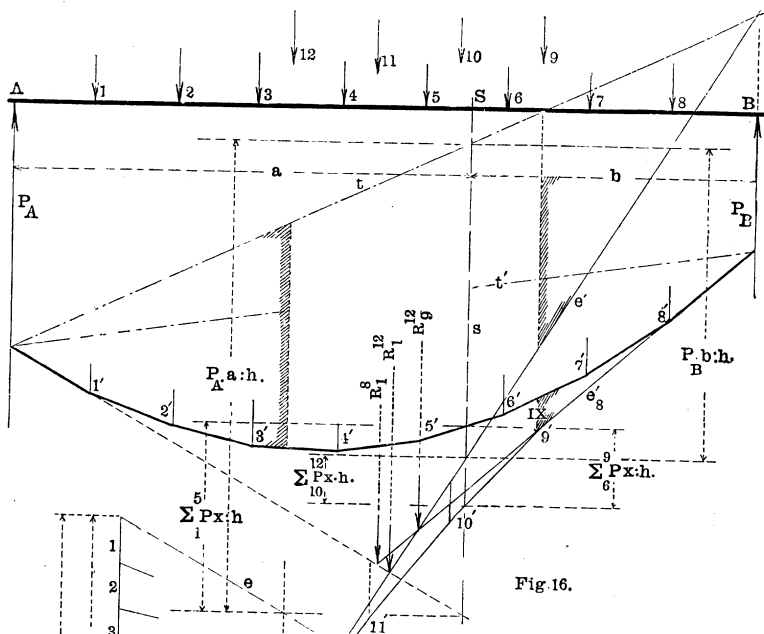
the line on which the bending moment ordinate is required. The bending moment here is

$$P_B \cdot b - (P_5 x_5 + P_6 x_6),$$

but on the line s , the ordinate from the point (s, t) , that is, the point where s and t intersect, (t is not shown in the figure but can easily be imagined) to the point (s, e') is $P_B \cdot b : h$ (art. 31 and fig. 13) where it has been shown that " $P_A \cdot a : h$ is measured by the intercept on the vertical s between the extreme ray e and the ray t of the cord polygon," and the ordinate from the point (s, e') to the intersection of s with $4'5'$ is only $P_5 x_5 : h$, and the ordinate from the point (s, e') to the intersection of s with $6'7'$ is $P_6 \cdot x_6 : h$, whence for the true measure of the bending moment we must deduct from the intercept on s between t and e' both the ordinate above e' to $4'5'$ and the ordinate below e' to $6'7'$.

With the regularly formed cord polygon the ordinate from (s, t) to $4'5''$ measures (31) the bending moment $: h$, and elementary geometrical considerations show that the intercept on s between $4'5''$ and $4'5' =$ intercept on s between e' and $6'7'$ which we will only indicate.—Breadth of triangles upon the same base VI , and between the same parallels VI and $77'$, at any given distance from the base are the same; again, breadth of triangle upon the same base V , and between the same parallels V and $77'$, at given distance from the base are the same.

II. *Several Interpolated Forces.*—In the same manner for many interpolated forces, as, for instance, from two series of downward



acting forces, in which the first may represent the weight of a girder supposed concentrated on its joints, and the second series the weight of one or more locomotives concentrated on their wheels.

Such an irregularly formed polygon is shown on fig. 16, for

twelve forces, four being interpolated where, as in the first case, the bending moment : h at any point S of the beam is obtained by deducting from the ordinate between the point (s, t) and the point (s, e') the ordinate on s intercepted between the lines $\overline{5'6'}$ and $\overline{9'10'}$. Fig. 16 has been so marked as to make this evident from a little consideration.

$$M : h = (P_B \cdot b - \sum_6^9 Px) : h = (P_A \cdot a - \sum_1^5 Px - \sum_{12}^{10} Px) : h.$$

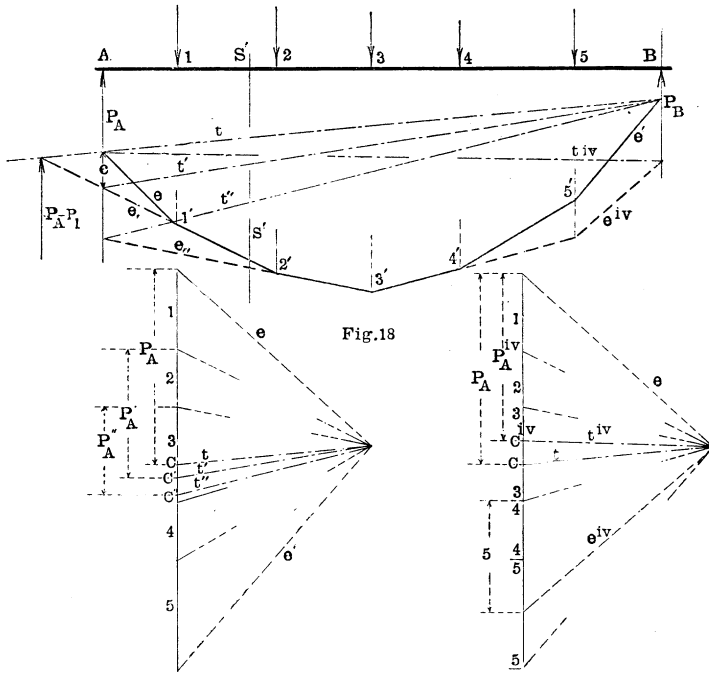
40. *Interpolating the Resultants of Additional Series of Forces in a Cord Polygon.*—The polygon of Case II. of last article is seldom required in its full extent, but when a second group of forces, as therein is superimposed on a beam, it is sometimes convenient to construct the cord polygon with the resultant of the second group instead of with the single forces. The ordinates of the cord polygon so constructed evidently give $M : h$ correctly at all sections outside of the group. The spaces within which $M : h$ can be correctly measured is lightly hatched in figs. 16 and 17. The ordinate IX . (fig. 16), equal to $\sum_7^8 Px : h$, being a negative ordinate. This proposition requires no formal demonstration.

41. *Alterations in Value of Bending Moment and Shearing Force arising from Taking Weights Off the Beam.*

I. *Beam fixed at one end.*—In this case the shearing force and bending moment being both obtained by purely additive processes, the abstraction of any weight P_n must diminish the shearing force on the further side by P_n itself, and the bending moment by $P_n x$, at any section S regarded as origin on the further side of P_n , the weights being numbered from the free end.

II. *Beam Supported at the Two Ends.*

(1) *Shearing force.*—Let fig. 18, P_1, P_2, \dots, P_5 be a series of weights upon a span AB with the usual force and cord polygons, P_A is the reaction of the left support equivalent to the shearing force up to P_1 , $P_A - P_1$, from P_1 to P_2 . Now, if P_1 be removed, the new reaction P'_A , equivalent to the shearing force from A to P_2 , is greater than the previous reaction to the right of P_1 and to the left of P_2 .



To obtain this new reaction we find the new closing line t' of the altered cord polygon and, transferring it to the new force polygon, $C'(1, 2)$, measures the new reaction P'_A , and it is evidently greater than $P_A - P_1$. If P_1 and P_2 be both removed the new reaction P'_A is obtained in the same way and is evidently greater than $P_A - (P_1 + P_2)$ which was the shearing force in front of P_3 before the removal of P_1 and P_2 .

A force as P_4 taken from amongst the other loads diminishes the reaction in front of the load for $\overline{U^{iv}(e, 1)}$ on force polygon which measures the new reaction is less than $\overline{U(e, 1)}$.

It is further evident that while one weight only, as 5 remains upon the beam, the reaction to the left is upward.

Whence we conclude, the shearing force in front of any load, as 3, is increased by removal of all weights, as 1 and 2, between it and the point of support A in front of it.

Bending Moments.—It is evident from a consideration of fig. 18 that the removal of any weight diminishes the bending

moment. The line t becoming t' , then t'' , gradually approaching the lower lines of the cord polygon. Certainly, the lower lines of the cord polygon retreat from the line t as t approaches them, but there is more deducted than added to the ordinates of the cord polygon.

Take, for instance, the lines t and t' , the side e becomes e_1 and t, t' , are the two sides of a triangle whose base is c ; e and e_1 are likewise two sides of a triangle whose base is also c ; but of two triangles upon the same base, and on the same side of it, at any given distance from the base, the breadth of the one having the least altitude is less than the breadth of the other, wherefore, &c.

42. Value and Line of Action of Resultant of Forces on One Side of a given Point in a Beam.—It is of fundamental importance to determine the resultant of all the forces upon one side of a cross-section of a beam, thus, figs. 18 and 13, to determine the situation and value of the resultant of all the forces to the left of S' , S' being situated between weights 1 and 2 in both figures.

The forces to the left of S' are $+P_A$ and $-P_1$ whence their resultant is $(P_A - P_1)$, whose line of action is thus found:—Produce the closing line t of the cord polygon to the left, produce also the side cut by the vertical through S' (in these figs. $\overline{1'2'}$) and the point of intersection of these two lines, t and $\overline{1'2'}$, is in the line of action of $(P_A - P_1)$, for $\overline{1'2'}$ produced becomes then the extreme ray e of both force and cord polygons, and the value of its bending moment at S' is the same, for it is measured by the same ordinate. For, calling S' the origin, $S'A = a$ and horizontal distance from S' to the vertical through the intersection of t and $\overline{1'2'} = a'$, we have

$$M : h = Pa : h - P_1x_1 : h = \text{intercept on } s'$$

$$\text{also} \quad M : h = (P_A - P_1)a' : h = \text{intercept on } s',$$

$$\text{also for } h : (P_A - 1) \text{ on force polygon} :: a' : \frac{(P_A - 1)a'}{h} \\ = \text{intercept on } s'.$$

In the same manner, had the breadth of our page permitted, we might have found a point on the line of action of

$$P_A - (P_1 + P_2)$$

by producing the side $\overline{2'3'}$ till it intersected t , and so on.

43. *The Bending Moment of all Forces upon a Continuation of the Beam is Negative.*—The intercept at any point D of the beam beyond the points of support between t and e produced is $-\frac{M}{h}$ for that point.

Section IX.—Turning Bending Moment Couples.

44. *Application of Turning a Couple to an Arched Beam.*—Let there be a given arched beam, fig. 20, $P_1, P_2 \dots P_6$ the vertical pressures, P_A the direction of the A abutment reaction, and A the point in the abutment through which this reaction goes, B the point in the corresponding abutment, and in the same horizontal with A through which the other reaction must go.

With a pole distance h form the usual force polygon and corresponding cord polygon, finding as formerly, by means of the closing line t , the vertical components P'_A and P'_B of the abutment reactions. The horizontal components H_A and H_B of P_A and P_B acting in the line AB must necessarily be equal and opposite. Through the point C draw a horizontal line meeting P_A in the force polygon parallel to P_A given in direction, in D , and from D draw P_B to the point $(6, e')$. This is the value and direction of P_B . In order to fix our attention, let us take any section S . Call S the origin.

There are three series of couples acting at the point S of the beam.

First and second series

$$P'_A \cdot a - (P_1x_1 + P_2x_2).$$

Third series

$$- H_A \cdot y_s.$$

y_s being the ordinate of the beam from AB at S , the bending moment of the beam is therefore

$$M_s = P'_A \cdot a - P_1x_1 - P_2x_2 - H_A \cdot y_s.$$

The sum of the first two series is measured as formerly (31).

In order to obtain $H_A \cdot y$ generally for all sections, proceed as follows:—

Lay off h vertically downward from the point C giving a point O' , join $O'D$. From the abutment point A , draw d' parallel to $O'D$, produce the vertical c through A upwards. Through the points 1, 2, 3 6 of the beam draw horizontal lines. The intercepts of these horizontal lines between c and d' measure $H_A \cdot y$ for the various points 1, 2, 3 of the beam, viz., $H_A \cdot y_1 : h$ for the point 1, $H_A \cdot y_2 : h$ for the point 2

Laying the values of these intercepts, vertically, in their respective places upon the cord polygon, we obtain a new polygon, marked by strong lines, and the new intercepts measure $M : h$ as it exists in the arch under the condition of P_A being given in direction, are measured within the shaded area.

In this figure, the bending moment of the arched beam is negative at 6, *i.e.*, it is being bent upward at that point.

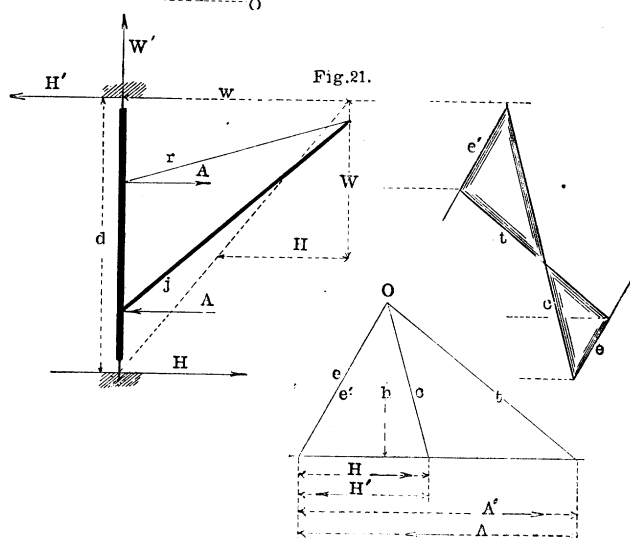
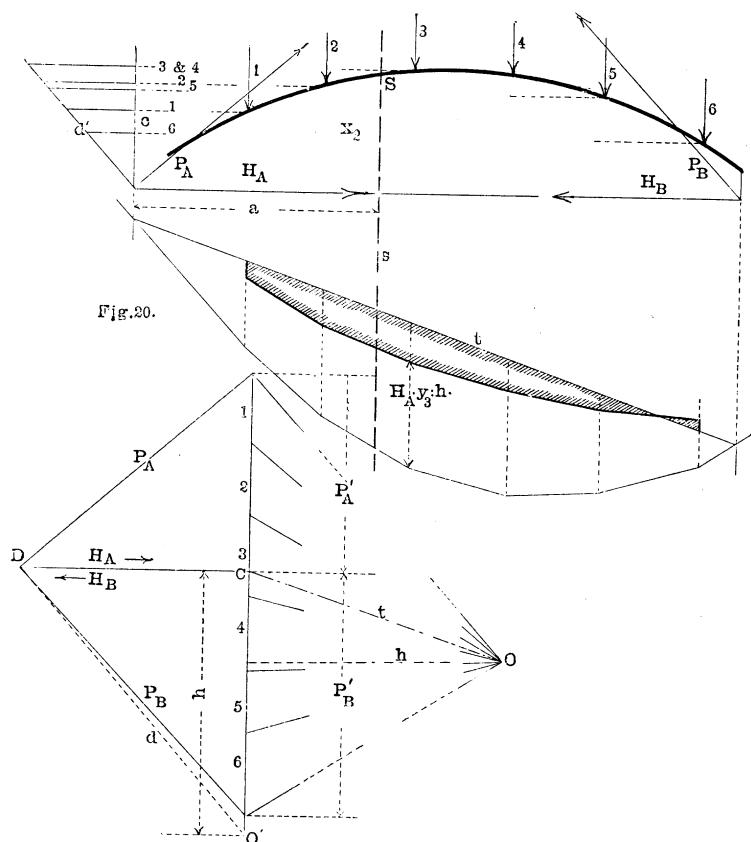
45. *Application of Turning a Bending Moment Couple to a Warehouse Crane.*—The impressed couple (fig. 21) consists of the vertical weight W and its reaction W' in the pillar of the crane, and arm w , resisted by the reaction couple consisting of the horizontal reactions H and H' through the top and bottom pivots and vertical arm d .

The impressed couple itself can only act on the pillar of the crane through the rod r and the jib j by means of the couple A and A' with arm a .

In the vertical of the weight W , and from the horizontal w through the upper pivot, lay off W . From the point (W, w) draw a line going through the lower pivot. From the lower end point of W draw a horizontal H meeting the previously drawn oblique line, that line H is the value of the force of the reaction couple $H \cdot d$ equal to and equilibrating the couple $W \cdot w$.

This couple $W \cdot w$, can, as already stated, only act through the couple $A \cdot a$.

Take then any pole distance h , and lay off horizontally the two supporting forces $+H, -H'$, the two extreme rays of the force polygon c and c' coincide, and we have the other ray c .



Draw in the corresponding rays e , c , e' in the cord polygon, supposing the forces $+A$ and $-A$ the supporting forces required, and inclose the cord polygon with the line t . Transfer t to the force polygon and we obtain the forces A and A' required, and

$$W \cdot w = H \cdot d = A \cdot a.$$

The cord polygon shows that the bending moment is of one sign in the upper part of the pillar and of the other sign in the lower part, and passes through the value zero, so that if the pillar yielded to its influence it would assume an \mathbf{Z} form.

Section X.—The Older Graphical Methods of Obtaining Bending Moment and Shearing Force, founded upon Analysis and the Hypothesis of Uniform Loading.

46. *Shearing Force Treated by the Older Graphical Method.*

I. *In the Case of a Beam Fixed or Inbuilt at One End, and Free at the other, fig. 22, and supposed Uniformly Loaded.*—Let

p be the load per unit of length

x the horizontal abscissa, the free end being origin

s the vertical ordinate

l the length of the beam

$$a = \frac{l}{2}$$

then

$$s = px.$$

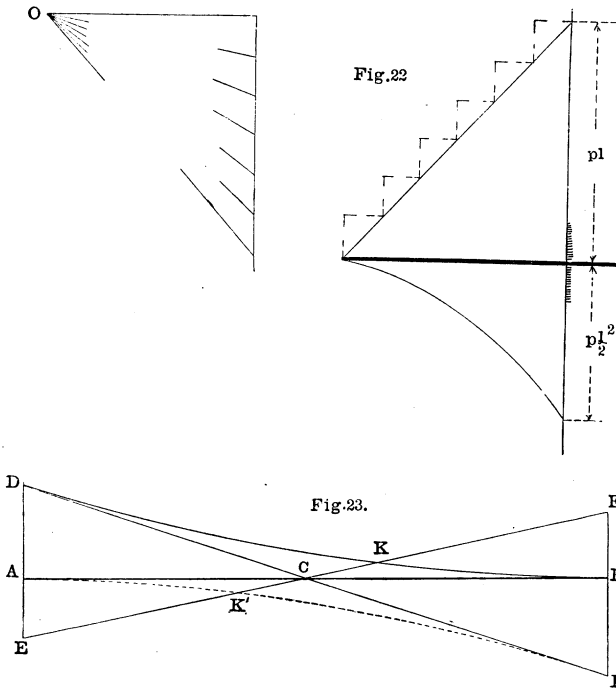
This is the equation to a straight line passing through the origin. At the fixed end, lay off $s = pl$, join the free end of this ordinate to the origin, and we obtain a graphical representation of the shearing force in this case.

II. *In the Case of a Beam Supported at Both Ends and Uniformly Loaded, fig. 23.*—Making the centre of the beam C the origin, then (46, I.) for a uniform load over the beam

$$s = \pm px.$$

At the two points of support, lay off as ordinates

$$s = +p\frac{l}{2} \text{ and } s = -p\frac{l}{2},$$



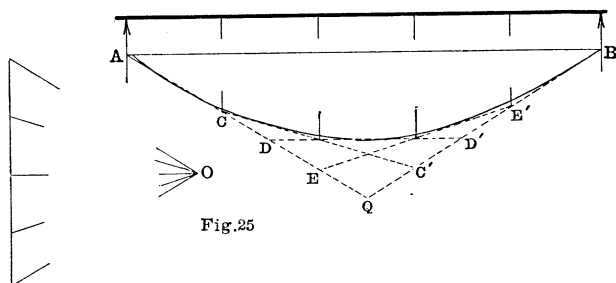
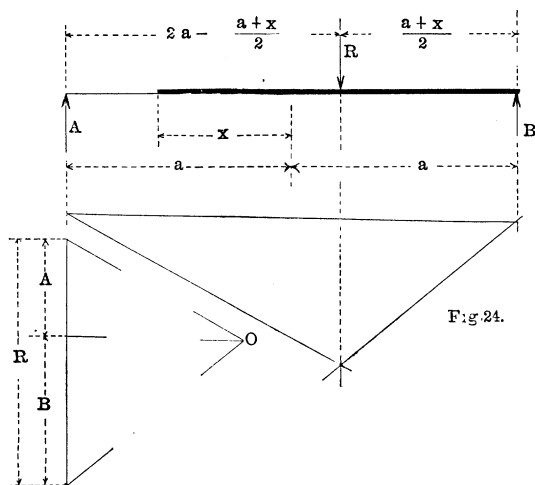
join the free ends D and D' of these two ordinates by a straight line necessarily passing through the origin C , and we obtain a graphical representation of the shearing force in this case.

III. *In the Case of a Travelling Load of Uniform Pressure per Unit of Length, required the Graphical Representation of the Shearing Force as it exists at any Section as the Load arrives at that Section.*—Let the load, fig. 24, arrive on the beam at support B moving forward to support A , at a certain instant it extends over the beam a length $a \pm x$ the centre of the beam being origin, the load therefore is

$$p = a \mp x$$

and the segments into which the resultant R of this load divides the beam, are

$$2a - \frac{a \mp x}{2} \text{ and } \frac{a \mp x}{2},$$



whence calling A and B the two reactions, we have the two following equations

$$A + B = p(a \mp x)$$

$$A(2a - \frac{a \mp x}{2}) = -B \cdot \frac{a \mp x}{2}$$

and reducing

$$S = A = p \cdot \frac{(a - x)^2}{4a}$$

the equation to a parabola.

47. *Bending Moment Treated by the Older Graphical Method.*—
I. *In the Case of a Beam Fixed at one End and Free at the other,*

fig. 22.—Under uniform loading our formula $\Sigma_1^n Px$ (28 and fig. 14) becomes

$$\Sigma_1^n Px = \int_0^x px = p \frac{x^2}{2}$$

the equation to a parabola. For

$$x = l, \quad p \frac{x^2}{2} = p \frac{l^2}{2}.$$

II. *In the Case of a Beam Supported at Both Ends and Uniformly Loaded, fig. 25.*—In this case the reaction at each point of support is

$$p \cdot \frac{l}{2}$$

placing the origin at A the bending moment at any abscissa x is

$$p \frac{l}{2} x - p x \frac{x}{2} = p \frac{x}{2} (l - x),$$

the equation to a parabola whose vertex is at the centre of the span.

48. *Change of Sign of Shearing Force in a Beam under Traveling Loads, taking the Weight of the Beam into Account.*—We must in railway girders consider the combined effect of a shearing force, arising from a constant load and that arising from a traveling load, both loads being considered uniform; the first load being that of the girder itself, the second that of a train in motion. We have seen that the shearing force arising from the first load, so conditioned, changes sign at the centre of the span, and is represented by a straight line DD' (fig. 23) crossing the span at centre, and whose extreme ordinates over the points of support A and B (46, II.) are

$$s = \pm p \frac{l}{2} = pa.$$

Considering $S \uparrow \downarrow P$ shearing as positive and $P \downarrow \uparrow S$ as negative, and drawing the straight line EE' (fig. 23) so that the positive or left hand ordinates are under the span line AB , and

let the travelling load come on from the right hand support B , then the shearing immediately in front of it is always positive; plotting this shearing *above* the span line we obtain the parabola BKD ; the vertical distance between these two lines EE' and BKD at any point x gives the *total* value of shearing at the instant when the travelling load has arrived at that point, the whole distance from B to the point x being loaded.

From B to K the positive shearing arising from the travelling load being less than that arising from the constant load, the shearing for that distance is evidently negative; from K to A positive.

Had the load come on to the span from the left hand, the shearing arising in front from it would always have been negative; plotting this shearing AD' below the span, then the total shearing from A to K' only would be positive; from K' to B negative; whence we see that *in a girder of uniform weight and uniform travelling load, which may approach from either end, there is a distance KK' on either side of the centre of the span subjected alternately to positive and negative shearing.*

When the travelling load lies upon the whole bridge then the vertical ordinates of the triangles EDC and $E'D'C'$ are the simultaneous values of the positive $S \uparrow P$ and of the negative $P \downarrow S$ shearing respectively.

49. Geometrical Determination of the Curves of this Section.—

In fig. 25, let the uniform load be supposed concentrated at equal distances along the beam, and form the force and cord polygons, then from elementary geometrical considerations

$$\frac{AQ}{BQ} = \frac{CD}{C'D'} = \frac{DE}{D'E'} \dots$$

wherefore (Proj. Geom.) $AQ, BQ, CC' \dots$ are tangents enveloping a parabola, and the more numerous the number of concentrated weights into which the uniform load is divided the more numerous the tangents, wherefore the curve may be made as near a parabola as we please.

The same reasoning applies to the parabolas of figs. 22 and 23.

CHAPTER II.

OPEN FRAME WORK.

Section I.—Extension of Force and Cord Polygon Theorems to Forces distributed in any manner in a Plane.

50. *Formation of Force and Cord Polygon to Forces distributed in any manner in a Plane.*—Let (figs. 26 and 27) 1, 2, 3, 4, 5, 6 be the lines of action of forces distributed arbitrarily in a plane. Let 1, 2, 3, 4, 5, 6 be their force polygon composed of lines parallel to the lines of action of the forces, and of lengths proportional to their force. The forces being supposed in equilibrium, this force polygon is a closed polygon.

From O (the point 1, 6) draw the diagonals $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ these diagonals are, by the theorem of the triangle of forces, respectively equal to the resultants of the forces after which they are named.

To form the corresponding cord polygon. Beginning with the forces 1 and 2, from their point of intersection draw the line $\overline{1, 2}$ parallel to $\overline{1, 2}$ in the force polygon till it meets the line of action of force 3, then from that point of intersection draw the line $\overline{1, 2, 3}$ parallel to $\overline{1, 2, 3}$ in force polygon till it meets the line of action of force 4 and so on.

These lines $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ are evidently the lines in which a cord, under the influence of the forces 1, 2, 3, 4, would be kept stretched, and the tensions along the different parts of the cord, are measured by the diagonals $\overline{1, 2}$, $\overline{1, 2, 3}$, $\overline{1, 2, 3, 4}$ of the force polygon.

In our figure, all these lines are in a state of tension, but we will, nevertheless, apply the name cord or link polygon to cases in which one or several, or all the lines of the imaginary cords or links are in a state of compression.

51. CULMANN'S THEOREM (51-53). *Axial Line and Reciprocity of Figure between a Combined Pair of Force and a Combined Pair of Cord Polygons, belonging respectively to the same Forces with One Displaced Force.*

I. *Their Axial Line*.—If we construct our cord polygon, beginning with the forces 2 and 3, instead of as before, with 1 and 2, we obtain a new cord polygon, fig. 27, in which the tensions along the new cord are measured by a new set of diagonals in the force polygon (shown dotted), viz. $\overline{2, 3}$, $\overline{2, 3, 4}$, $\overline{2, 3, 4, 5}$.

This new cord polygon and the one previously constructed are connected by the geometrical property that the resultant forces, viz. :

$\overline{1, 2}$ on the first, intersects $\overline{2}$ on the second or new cord polygon

$\overline{1, 2, 3}$ „ „ $\overline{2, 3}$ „ „ „

$\overline{1, 2, 3, 4}$ „ „ $\overline{2, 3, 4}$ „ „ „

$\overline{1, 2, 3, 4 \dots n}$ „ $\overline{2, 3, 4 \dots n}$ „ „

in the line of action of the force 1.

This is capable of an elementary demonstration of which we will only indicate the steps by help of fig. 27a.

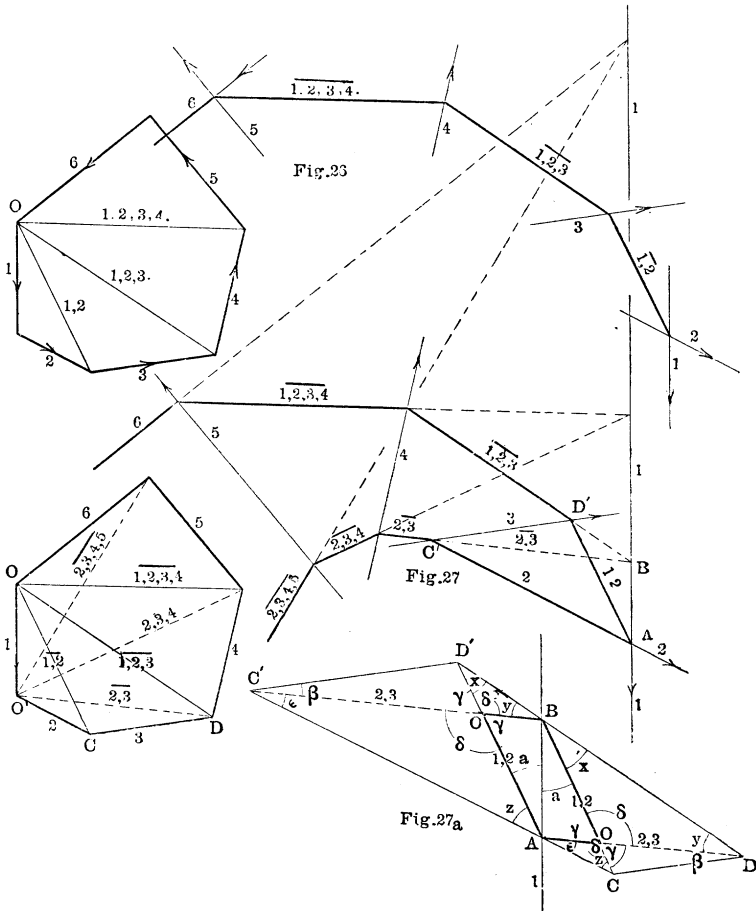
Let AB , fig. 27a, represent force 1 in direction, and describe on it part of the force polygon, viz., the quadrilateral $\overline{1, 2, 3}$ and $\overline{1, 2, 3}$, and its diagonals $\overline{1, 2}$ and $\overline{2, 3}$. On the other side of AB continue $\overline{2}$, and draw the diagonals $\overline{1, 2}$ and $\overline{2, 3}$ from the opposite points of AB , and through C'' where $\overline{2}$ and $\overline{2, 3}$ intersect, draw $3'$ parallel to $\overline{3}$ cutting the diagonal $\overline{1, 2}$ in D' . Join $D'B$, then $D'B$ will be in the same straight line as BD , that is, as $\overline{1, 2, 3}$.

This figure $AC'D'B$ is evidently a portion of the combined cord polygons, fig. 27, which have for comparison been similarly marked.

From the following elementary principles, first, that parallel lines make equal angles with the same line; second, that the alternate angles made by parallel lines cutting a line are equal; third, that alternate angles are equal, we obtain in the figure

$$\angle \alpha = \angle \alpha', \quad \angle \beta = \angle \beta' \dots$$

From these equalities we find that $AOBO'$ is a parallelogram, and that AOC , $AO'C'$; COD , $C'O'D'$, are two pairs of similar triangles, we find from these that BOD , $B'O'D'$ are similar



triangles, that therefore $\angle x = \angle x'$ and $BO, D'O'$ being parallel, $D'BD$ form a straight line.

It follows hence, that $AC'BD'$ in the cord polygon, fig. 27, being a similar figure to $AC'BD'$ in fig. 27a, the tension $\overline{2, 3}$ of the second cord polygon meets the tension $\overline{1, 2, 3}$ of the first cord polygon, in the line of action of the force 1.

Taking in force polygon, the quadrilateral formed by the forces $\overline{1, 2, 3, 4}, \overline{1, 2, 3, 4}$ and its diagonals $\overline{1, 2, 3}$ and $\overline{2, 3, 4}$, a similar method of demonstration would prove that the tension line $\overline{2, 3, 4}$ of the one cord polygon meets the tension line

$\overline{1, 2, 3, 4}$ of the other cord polygon in the line of action of force 1.

It can likewise be easily perceived that this property of the combined cord polygons can be proved on statical principles.¹

II. *Their Reciprocity of Figure.*—It will be observed that these two quadrilaterals $ABCD$, $ABC'D'$, are not *similar* figures in the ordinary acceptation of the word, but *reciprocal*, that is, have not only the lines of the one parallel to the lines of the other, but a triangle in the one corresponds to a node in the other, thus the triangle in the force polygon whose sides are 1, 2, and $\overline{1, 2}$ corresponds to the node A in the cord polygon, where meet the lines 1, 2, and $\overline{1, 2}$, parallel each to each with the sides of the triangle, the triangle in the force polygon 1, $\overline{2, 3}$, and $\overline{1, 2, 3}$ to the node B in the cord polygon where meet the lines 1, $\overline{2, 3}$, and $\overline{1, 2, 3}$.

From thence we might go on to the reciprocity of the similarly named quadrilaterals 1, $\overline{2, 3}$, $\overline{4}$ and 1, $\overline{2, 3, 4}$ and its diagonals $\overline{1, 2, 3}$ and $\overline{2, 3, 4}$, and so forth.

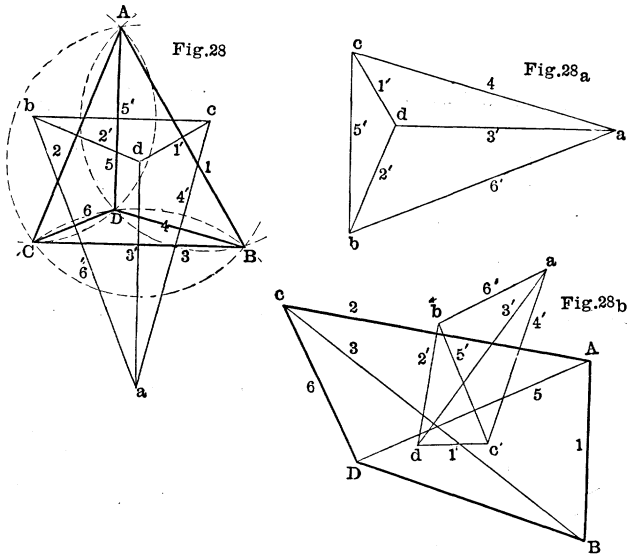
Whence it follows that the combined force polygons and combined cord polygons are reciprocal.

52. *Elementary Demonstration of Reciprocal Quadrilaterals.*—The possibility of forming a reciprocal quadrilateral is capable of a very elegant yet elementary demonstration.

Being given a figure formed of six lines joining four points in a plane, it is always possible to construct a second figure, equally composed of six lines joining four points, and such, first, that each side of the one figure corresponds to a side of the other, parallel or perpendicular to or making a given angle with the first. Second, that all systems of three concurring lines in the one correspond to three sides of a triangle in the other. The two figures satisfying these conditions are said to be reciprocal.

Let $ABCD$, fig. 28, be the figure given. It comprehends
 $\begin{array}{cccc} & d & c & b & a \\ & & & & \end{array}$
 four triangles ABC , ADB , ADC , BDC . The figures formed by

¹ See Collignon, *Cours de Mécanique*, première partie, p. 272. Paris, Dunod, 1877.

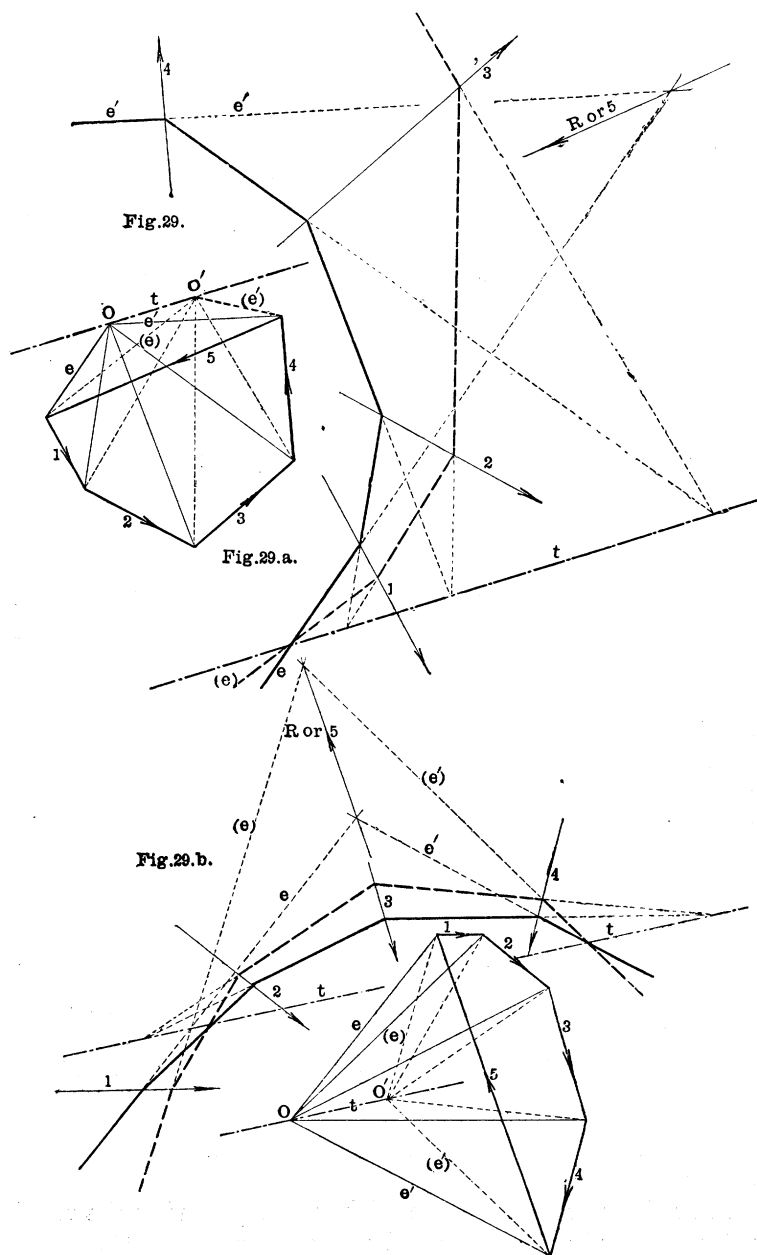


the six lines joining the centres d, c, b, a of the circles circumscribed to these triangles, evidently possess the properties enunciated. This results from the theorem that the perpendiculars raised upon the middle points of three sides of a triangle concur in a point. We will not proceed with a formal demonstration.

1. To all lines 1 of the given figure corresponds a line 1' of the new figure and perpendicular to the line given. 2. To all systems of three lines, as 1, 4, 5, forming a triangle in the given figure corresponds a node of three lines concurring in a point, 1', 4', 5' in the new figure, and therefore inversely. If we turn either of these figures round through a right angle, the lines in the one will then be parallel to the lines in the other (fig. 28a).

It does not affect this proposition that, say, the two triangles ACD, BCD , fig. 28b, should be so distorted from what we may call their normal form in fig. 28, as to be as it were folded, the one over the other, though when thus or otherwise deviating from the normal form, the demonstration cannot be so clearly perceived.

From this demonstration we might again proceed to the proof of the combined pair of force and cord polygons of art. 51.



53. *Axial Line and Reciprocity between the combined Pair of Force and combined Pair of Cord Polygons to the same Forces, Pole being displaced.*—The homologous links of two cord polygons to the same forces, derived respectively from two force polygons, whose poles are O and O' , meet on a line t parallel to the line OO' (figs. 29a and 29b).

For we may regard the given forces as $e, 1, 2, 3, 4, (e'), t$, in the first pair of force and cord polygons and as $t, e, 1, 2, 3, 4, (e')$ in the other, and the proposition is reduced to that of art. 51.

54. *Resultant and Line of Action of Resultant of Forces in a Plane.*—Let (fig. 29) 1, 2, 3, 4 be the given forces in direction and magnitude in the force polygon, and likewise in position on the plane of the paper.

I. *Value of Resultant.*—In the force polygon join the free ends of 1 and 4 by the line 5, then 5 is the resultant in direction and magnitude.

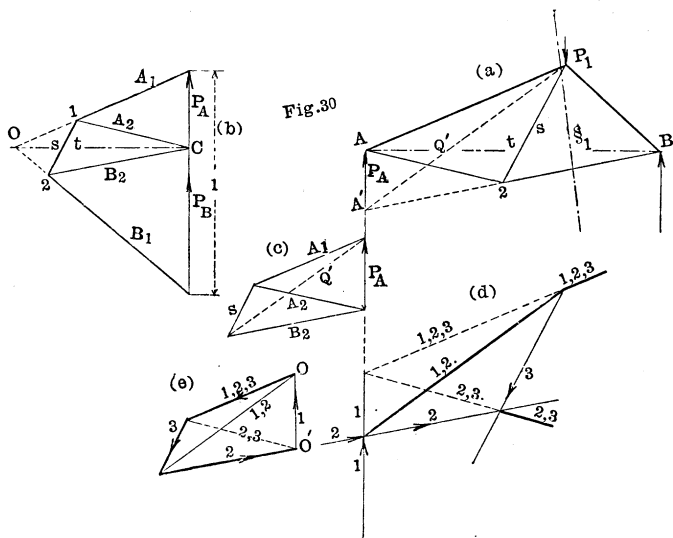
II. *Line of Action of Resultant.*—Take any pole O for the force polygon, and form the corresponding cord polygon by means of art. 50. Extend the extreme links e and e' of the cord polygon till they intersect in (e, e') a point in the line of the resultant R . Through this point draw R parallel to R or 5 in the force polygon.

For it is evidently the resultant of the two tensions e and e' , and consequently with opposite sign equilibrates the action of the forces 1, 2, 3, 4.

Section II.—Application of preceding Force and Cord Polygon. Theorems to Determination of Tensions or Stresses in the Bars of Framework.

55. *Condition of Indeformability in Framework.*—Framework must possess the property of indeformability under the action of applied forces, and, in order to satisfy this condition, it must necessarily be divided into triangles.

56. *Example of Irregular and Simple Framing.*—Fig. 30a is an example of an irregular and simple frame $A, 1, B, 2$. The impressed forces are the weight 1 acting downwards at node or joint 1, and its two reactions P_A and P_B at the points of support.



These reactions are obtained after articles 20 and 21 (figs. 11 and 13). In this simple case we can consider $A1B$ as the cord polygon, AB its closing line t , O is the pole of the force polygon, fig. 30(b), formed from it. Force 1 is R of fig. 11, and OC or t drawn parallel to AB or t divides 1 or R into P_A and P_B in the point C . From C draw $\overline{A2}$, $\overline{B2}$, parallel to $A2$, $B2$ in the frame, meeting $A1$, $B1$ in the points 1 and 2; P_A , $A1$, $A2$ are in equilibrium, for $A1$ and $A2$ in the frame alone can transmit any pressure to A , and $A1$, $A2$ in the force diagram measure the tensions in the bars $A1$ and $A2$ in the frame diagram. In the same manner $B1$ and $B2$ in the force diagram measure the tensions in the bars $B1$ and $B2$ in the frame diagram.

Now (by art. 22) e , and e' would measure the stresses in the bars $A1$, $B1$ were the bars $A2$, $B2$ not in the frame, but owing to their presence, the bars $A1$, $B1$ are less in stress by the quantities $O1$, $O2$ respectively. The resultant of $O1$ and $O2$ is s in the force diagram.

But we will employ a method more general to the solution of this and other examples.

I. *Transformation of the Frame.*—In the frame diagram produce $B2$ till it intersect the line of action of P_A in A' . Join the

node 1 and the point A' by a line Q' , and let the frame now be transformed to $A'1B$.

Notice (1) P_A and P_B are independent of the form of the frame.

(2) The transformed frame is a triangle in equilibrium or a cord polygon (fig. 11) of which BA' is the closing line.

(3) The stresses in P_A , Q' and BA' form a triangle of forces in equilibrium of which one P_A is given and the directions Q' and BA' of the other two.

The stresses in P_B , $B1$, BA' form a triangle of forces in equilibrium of which one, P_B is given and the directions $B1$ and BA' of the other two.

(4) The stresses in Q' and BA' and in $B1$ and BA' depend solely, *first*, on P_A and P_B , and *second*, on their own direction.

(5) The stress in BA' may be regarded, *first*, as the resultant of P_A and stress in Q' , *second*, as the resultant of P_B and stress in $B1$.

II. *Original Frame*.—Seeing that the stress in $B2$, that is BA' , depends only on P_B and the direction of $B1$, its value remains unchanged, whilst for Q' we substitute the original bars $A1$ and s , whence, as in either case the node 1 is in equilibrium, the stresses in $A1$, s and Q' form a system in equilibrium.

Again, as the influence of the stress in $B2$ was communicated to Q' through BA' , (BA' and P_A holding Q' in equilibrium) so its influence must be communicated to $A1$ and s through the bar $A2$, and the stresses in $A2$, s and $B2$ form a system in equilibrium.

III. *Notation of Art. 51*.—Employing now the notation of arts. 50 and 51, for the forces and tensions

$$\begin{array}{rcl} P_A & \text{being called} & 1 \\ \overline{BA'} \text{ or } \overline{B2} & \text{,,} & 2 \\ s & \text{,,} & 3 \end{array}$$

Then from the previous elucidation it follows that

$$\begin{array}{rcl} Q' & \text{shall be called} & \overline{1, 2} \\ A2 & \text{,,} & \overline{2, 3} \\ A1 & \text{,,} & \overline{1, 2, 3} \end{array}$$

being the resultants of the forces after which they are now named.

In fig. 30*d* the frame is shown with this notation, and we see that we have the two combined cord polygons of art. 51, whose forces differ by the force 1, giving a figure reciprocal to their combined force polygons (fig. 30*e*), whence s in the force diagram is parallel to the bar s .

The determination of the stresses in the bars resolves itself, therefore, into finding the reciprocal combined force polygons of the combined cord polygons, composed of the actual and ideal frames. This requires no further elucidation.

57. *Definition of Frame Diagram and Force Diagram.*—We shall in future name the skeleton drawing of a frame, a frame diagram, and the force polygon of its tensions or stresses a force diagram, in order to distinguish them from the subsidiary cord and force polygons of Chapter I.

58. *Method of Sections, assisted by Line of Action of Resultant of Forces on One Side of Section.*—Referring to the example in

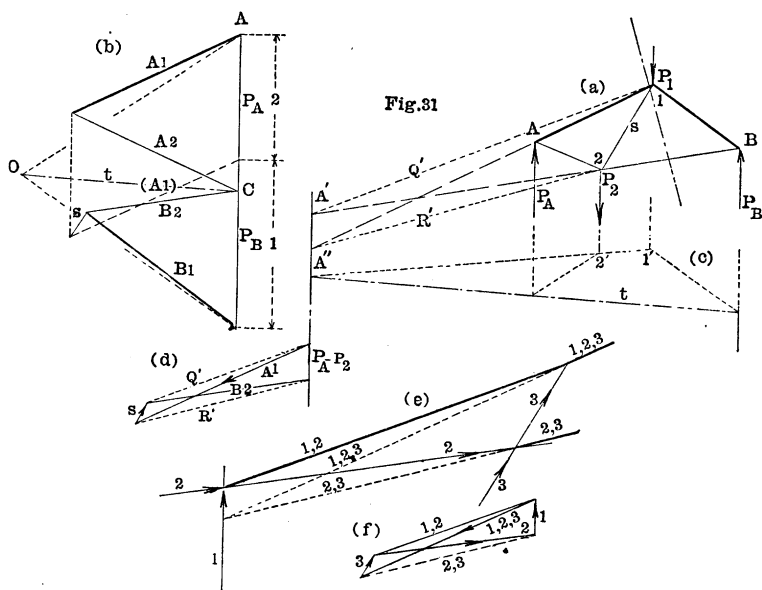


fig. 30 which we have analysed in art. 57. Let us suppose a section (§₁) through the frame on the left side of force and node 1.

Now in 56 we have seen that

(1) P_A on the left of this section equilibrates all the forces to the right (in this example P_B and 1.)

(2) That the stress in $B2$ can be made to depend on data all to the left of this section, viz., on P_A , its own direction and direction of Q' (56, 4), whence we can solve the three stresses, viz., those in $A1$, s and $B2$ without any reference to the form of the framework to the right hand of the section, (§₁). This gives the method of sections a generality in application, which we can best explain by an example not quite so simple as fig. 30.

Fig. 31*a* is a frame diagram similar to fig. 30*a*, but loaded on both joints 1 and 2.

(1) Form with any pole O a force and cord polygon (figs. 31*b*, 31*c*) and by means of the closing line t obtain the two reacting forces P_A and P_B .

(2) Conceive a section (§₂) to the left of joint 2. The value of stress in $A2$ and $A1$ depends entirely on P_A and the directions of the bars $A2$ and $A1$, whence the triangle of forces, P_A , $A2$ and $A1$ on the force diagram.

(3) Conceive a section (§₁) to the left of joint 1, then the forces to the left are $+P_A$ and -2 , and we find by method of art. 42 the line of action of the resultant, viz., of $+P_A - 2$, and shown upon fig. 31*c*.

(4) Extend $\overline{B2}$ to the line of action of $P_A - 2$, and draw in the imaginary bar Q' . Conceive as formerly the frame transformed to BA' , Q' and $B1$. BA' is under the action of the same forces as formerly $B2$ was, for it depended on P_B , the direction of $B1$, and its own direction. Now as the forces in the transformed frame are the same as those in the original frame (for P_A and P_2 have only been substituted their resultant $P_A - 2$ in its line of action), the value of P_B has not been altered, nor the directions of $B1$ and $B2$; its value therefore has

not been changed. We find, therefore, the stress on $B2$ by means of a triangle of forces formed by $P_A - P_2$ and the directions of $B1$ and $B2$.

Again, by taking a section (§2) close to the left of 2, and extending $A1$ to the line of action of $P_A - P_2$ in the point A'' and joining $A''1$ by the imaginary bar R' , we do not alter the value of the stress in $A1$, for the turning action of P_A around node 2 is $P_A \cdot a$, and of P_2 is $P_2 \cdot 0$, but $(P_A - P_2)$ acts in the resultant of P_A and P_2 , whence $(P_A - P_2)x = P_A \cdot a$, whence by means of a triangle of forces formed by $P_A - P_2$, and the directions of R' and $A1$, $A1$ can be determined.

In the actual frame, the turning action of $A1$ is equilibrated by $\overline{A2}$, in the ideal case by R' , and as $P_A \cdot a = (P_A - P_2)x$ it follows that, as we otherwise know, $A1$ and $A2$ can be determined by the triangle of forces formed by P_A and the directions of $A1$ and $A2$.

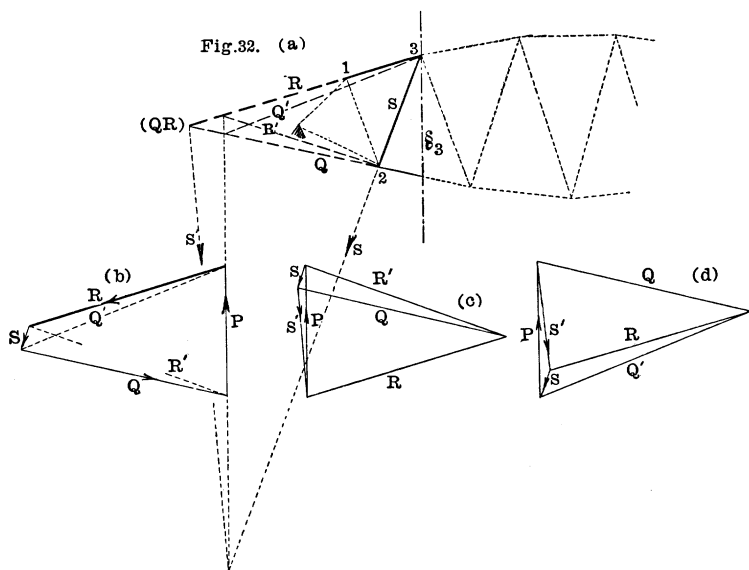
It is unnecessary to find $A1$ by means of the link R' , but R' gives us the remaining line of two cord polygons differing by one force, whose homologous sides meet in the line of action of the additional force, and, forming their reciprocal figure, we find the stresses in the bars $A1$, $A2$, s .

Calling after 49 and 50, in fig. 31e, 31f,

| | | | |
|-----------------------------|---|---|----------------------|
| Forces | $P_A - P_2$ by the symbol 1 | | |
| Bar and stress in bar, $B2$ | „ | „ | 2 |
| „ „ „ s | „ | „ | 3 |
| then | Q | „ | $\overline{1, 2}$ |
| „ | R' takes the symbol $\overline{2, 3}$ | | |
| „ | $A1$ | „ | $\overline{1, 2, 3}$ |

and we can at once recognise the combined cord and force polygons of art. 51, fig. 27.

59. *Method of Sections with Line of Action of Resultant on One Side of Section, continued.*—This method is perfectly general. Let us now take a more complex frame diagram, (figs. 32, 33, 34, 35) and let there be found first, by means of force and cord polygon, and proposition of art. 42 (these polygons are not given in figure), the value and position of all the forces acting beyond



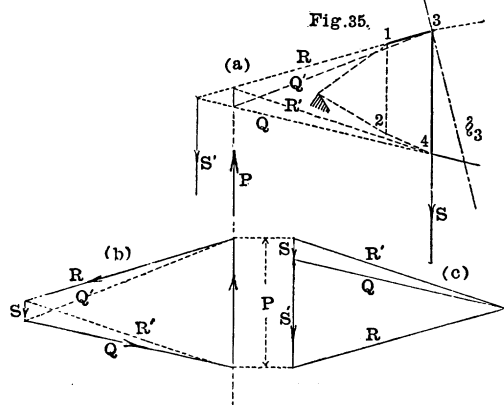
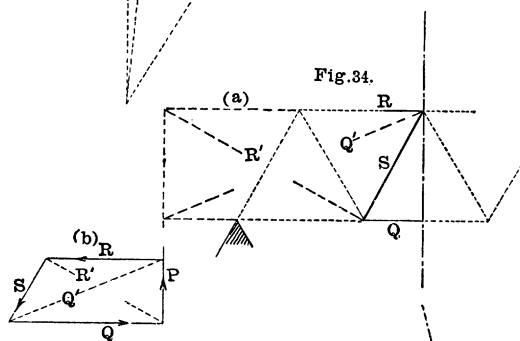
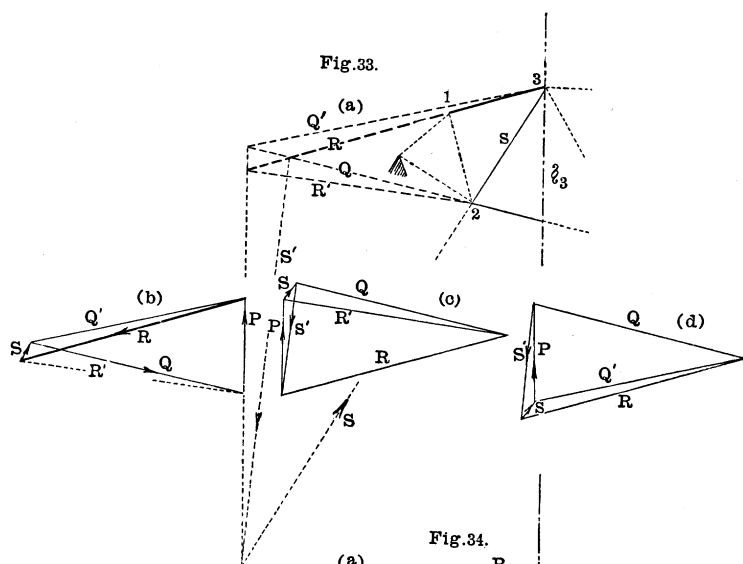
a given section, for instance to the left of S_3 , fig. 32, that is, let, as formerly, P_A and P_B , the reactions of the two points of support, be found, then at S_3 , $P = P_A - (P_1 + P_2)$, (P_1 and P_2 acting at the joints 1 and 2 respectively), and equal but opposite to the sum of the remaining forces to the right of S_3 .

The three bars of the frame R , s , and Q hold this in equilibrium, and if, as in (58) we draw in the imaginary bar Q' meeting Q produced in the line of action of P , and conceive the frame transformed so as to have the point (Q, Q') for the point of support, then $P \cdot x = P_A \cdot a - (P_1 a_1 + P_2 a_2)$, the turning moment being resisted by Q , either in the transformed frame or in the original frame, whence P , the stresses in Q and in Q' are in equilibrium and form a triangle of forces.

For Q' we may now substitute R and S (56, II.) which form with Q' therefore a system in equilibrium, wherefore stresses in Q' , s , and R , form a triangle of forces.

In the same manner, by taking a section to the left of joint 2 we might find R' , and we have again the combined cord polygon, whose combined force polygon is a reciprocal figure.

We can thus proceed from section to section and determine the stresses on all the bars.

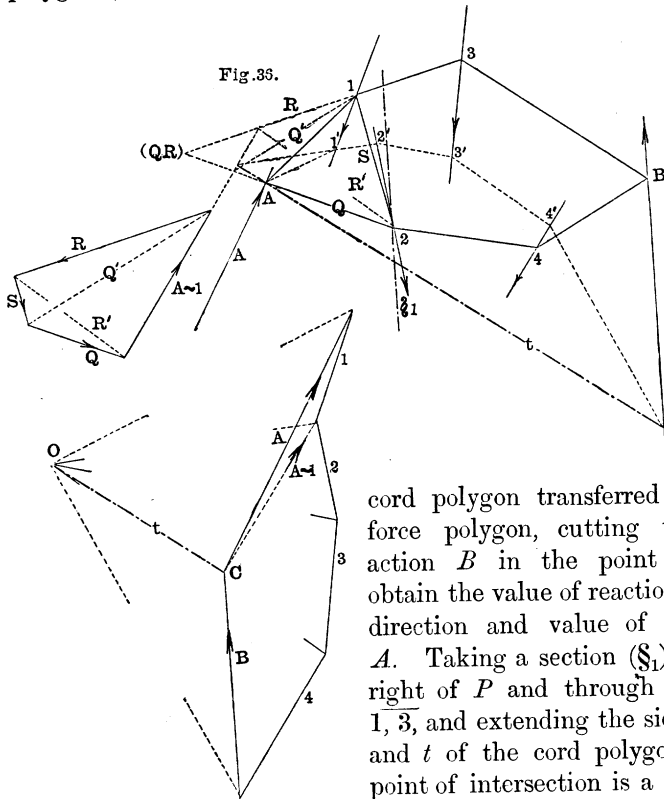


60. *Variations in Form of the Force Diagrams.*—The force diagrams, fig. *c* and *d*, represent another method of obtaining the strains in *Q*, *R*, and *s*, by resolving *P* into two forces *S* and *S'*, whose directions are given. *S'* going through the point (*Q*, *R*) must equilibrate *Q* and *R*.

These forms of force diagrams are in many circumstances more convenient than the previous forms, figs. *b*.

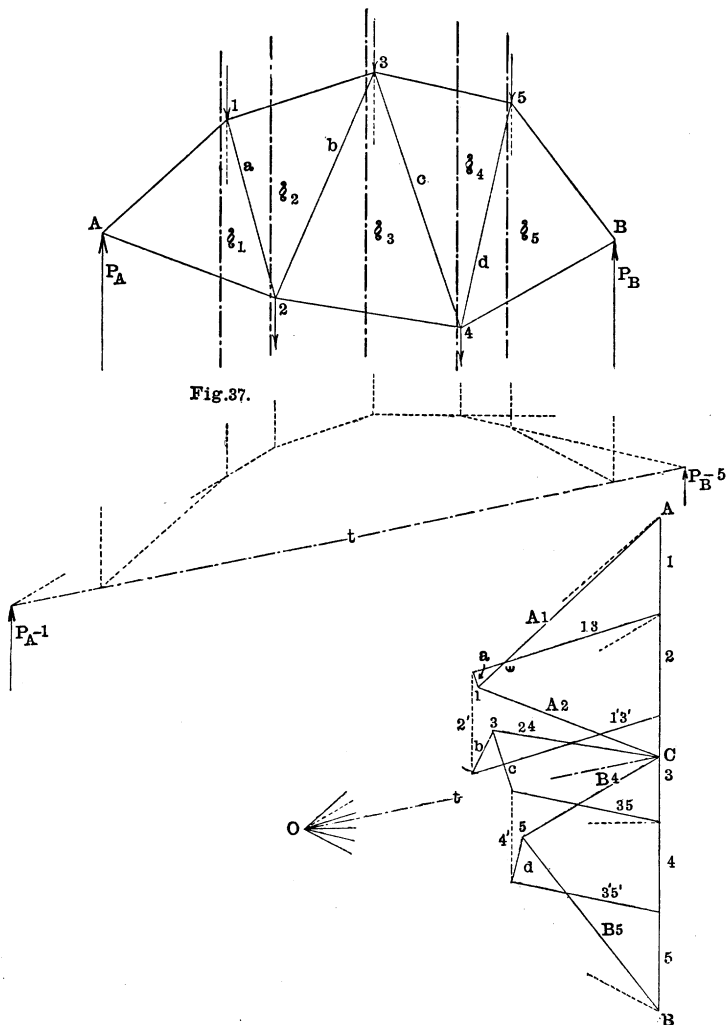
61. *Method of Sections, with Line of Resultant of Forces on One Side of Section, applied to Forces acting upon the Joints of a Frame in any direction.*—Fig. 36 is a frame of perfectly general form, with forces acting on its nodes in arbitrarily given directions, and the direction of one reaction, say *B*, given.

With any pole *O* form a force polygon and corresponding cord polygon (50 and 51), and by means of the closing line *t* of the



cord polygon transferred to the force polygon, cutting the reaction *B* in the point *C*, we obtain the value of reaction *B* and direction and value of reaction *A*. Taking a section (*S*₁) to the right of *P* and through the bar 1, 3, and extending the sides 1' 2' and *t* of the cord polygon, their point of intersection is a point in

the line of action of $A \hookrightarrow 1$, and joining the points C , and $(1, 2)$ in the force polygon, we obtain the direction and value of $A \hookrightarrow 1$. Through the point in the line of action of $A \hookrightarrow 1$ just found, draw a line parallel to the direction of $A \hookrightarrow 1$ as found in the force polygon. From the joint 1 of the frame we can draw in the imaginary link Q' meeting Q produced in the line of action of $A \hookrightarrow 1$, we have now in the frame diagram, as formerly, the combined cord polygons of which the combined force polygon is



a reciprocal figure, and furnishes, as formerly, the stresses in the bars $\overline{1, 3}$, s and Q .

In this manner we can proceed section by section for every other group of three bars.

The demonstration in art. 59, being general, completely includes this case, remembering that the lever arms of a turning force around any node, are measured by a perpendicular to the line of action of that force let fall upon it from the node.

62. Cases in which Force and Frame Diagrams are Similar Figures.—The force and frame diagrams are similar figures when the force and the bar s are parallel; fig. 35 illustrates this for the ordinary case of a vertical force and vertical bar.

63. Method of obtaining Force Diagram by Method of Sections, independently of Line of Action of Resultant of Forces on One Side of Section under consideration.

I. Frame with Simple Nodes.—Fig. 37 is a perfectly general and irregular framework irregularly loaded at the joints 1, 2, 3, 4, 5; the stresses on the bars are required.

By means of a force polygon, with any pole O and its corresponding cord polygon, we find the closing lines t , and thence the two reactions P_A and $P_B = CA$ and BC respectively.

Beginning now at the point A and working towards the right, consider first a section (\S_1) passing to the left of 1 and through $A2$.

The reaction $P_A = CA$, the stresses in $A1$ and $A2$ are in equilibrium, for, $A1$ and $A2$ are the only members which transmit a force to the point A .

On AC describe the triangle of forces required by this system.

Consider \S_2 passing to the left of joint 2 and through the bar $\overline{1, 3}$.

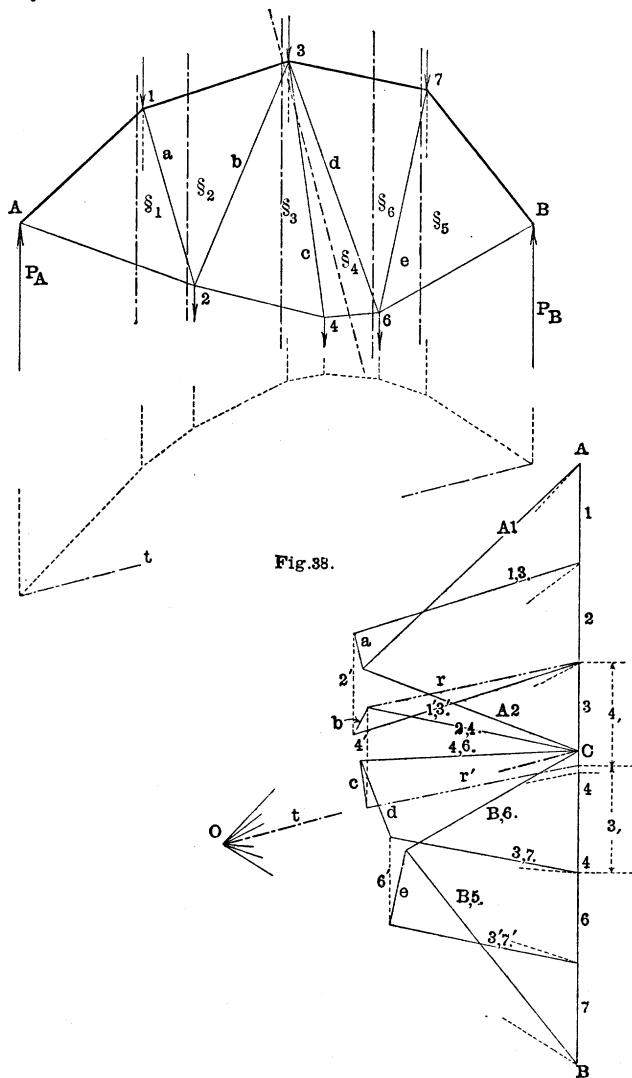
The reaction there is $P_A - 1 = \overline{C(1, 2)}$, and this is held in equilibrium (fig. 33) by S' and S , that is by S , Q , and R .

In force and frame diagrams, fig. 33

| | P | Q | R | S |
|-----------------|----------------------|-------|--------------------|--------------------|
| In this diagram | $\overline{C(1, 2)}$ | $A2$ | $\overline{1, 3}$ | a |
| Of these are | Given | Given | Req ^d . | Req ^d . |

Wherefore, from the extremity 1 of $\overline{A2}$, draw a parallel to bar a and through the point (1, 2), draw $\overline{1, 3}$ parallel to bar $\overline{1, 3}$ meeting a just drawn.

We thus obtain the closed polygon $C(1, 2), \overline{A2}, a, \overline{1, 3}$, whose sides measure respectively, the stresses in the bars having the same symbols.



Consider \S_3 to the left of joint 3 and through the bar $\overline{2, 4}$.

The reaction is now $P_A - (1 + 2) = \overline{C(2, 3)}$, held in equilibrium by S and S' as formerly.

In force and frame diagrams, figs. 32 or 33

$P \qquad R \qquad Q \qquad S$

In this frame diagram

$\overline{C(2, 3)} \qquad \overline{1, 3} \qquad \overline{2, 4} \qquad b$

Of these are

given given req^d. req^d.

wherefore from the point (2, 3) draw $\overline{1', 3'}$ equal and parallel to $\overline{1, 3}$. From its extremity draw b parallel to bar b , meeting the ray $\overline{2, 4}$ from C parallel to bar $\overline{2, 4}$ in point 3.

We thus obtain the closed polygon $\overline{C(2, 3)}, \overline{1', 3'}, b, \overline{2, 4}$, whose sides measure the stresses in the bars having the same symbols.

Consider \S_4 to the left of joint 4, the reaction is now $P_A - (1 + 2 + 3)$ and has become negative and is therefore measured from C downward, and is thus measured by $\overline{C(3, 4)}$.

The same process is repeated until the line $B5$ has been drawn from the point of 5, when, if the drawing has been accurately done, the line $B5$ will go through the point B .

In the same manner, by taking sections to the right of the joints we might begin at B and end at the point A .

It may be made a matter of observation, that for every node or joint in the frame diagram there is a reciprocal closed polygon in the force diagram. Take, for instance, the node 2, from whence in the framework issue the lines, force 2, bars $\overline{A2}, a, b, \overline{2, 4}$. In the force polygon, force $2', b, \overline{24}, \overline{A2}, a$.

II. *Frame with a Compound Node.*—If three connecting bars unite in the same joint as in fig. 38, where the three bars b, c, d unite in the same joint 3 the determination of their stresses is worthy of notice.

Beginning at A proceed, as in last case, until we obtain the stresses on bars $\overline{1, 3}$, and b . From the point (2, 3) force diagram suppose the line r drawn (it need not be drawn); r is the resultant of the stresses in the bars $\overline{1, 3}$ and b .

Consider now \S_4 , the reaction is $P_A - (1 + 2 + 4)$, wherefore, from the point (2, 3) lay off force 4_1 downwards, and through its

lower end point draw r' equal and parallel to r , from thence draw c meeting the ray $\overline{4, 6}$, or formulating

Force and frame diagrams fig. 32

| | | | | |
|--------|-----|-----|-----|-----|
| Forces | P | R | Q | S |
|--------|-----|-----|-----|-----|

In fig. 38

| | | | |
|---------------|------|-------------------|-----|
| $C(4_1, 3_1)$ | r' | $\overline{4, 6}$ | c |
|---------------|------|-------------------|-----|

of these are

| | | | |
|-------|-------|--------------------|------|
| given | given | req ^d . | req. |
|-------|-------|--------------------|------|

whence the construction described.

It will be observed that in this case it is no longer the resultant of the two booms R and Q which forms S' , but the resultant of R , b , Q .

Consider now §6. The reaction is $P_A - (1 + 2 + 3 + 4)$ and formulating

Force and frame diagrams, fig. 32

| | | | | |
|--------|-----|-----|-----|-----|
| Forces | P | Q | R | S |
|--------|-----|-----|-----|-----|

In fig. 38

| | | | |
|-------------|-------------------|-------------------|-----|
| $C. (4, 6)$ | $\overline{4, 6}$ | $\overline{3, 7}$ | d |
|-------------|-------------------|-------------------|-----|

Through the point $(4, 6)$ draw $\overline{3, 7}$ parallel to bar $\overline{3, 7}$, and through point $(c, \overline{4, 6})$ draw d parallel to bar d .

Proceed as in first case to the end.

It will be observed that in these two general frames the joint 1 on the upper bar or boom has only one connecting bar a attached to it, whilst the joint 2 on the lower bar or boom has two bars a and b attached. If the arrangement of connecting bars is such that the joint 1 has two bars, whilst the joint 2 has one bar, as in fig. 39, then the mode of procedure is modified. For instance, in considering joint 1, instead of determining the stresses by means of the three bars $\overline{1, 3}$, a , $\overline{4, 2}$, we determine it by means of $\overline{A1}$, $\overline{2, 4}$, a . Fig. 39 in the next article is an example.

64. *Method of Sections continued, applied to the Construction of Partial Force Diagrams, in order to determine the Stresses in connecting Bars when in front of an advancing Travelling Load.*—Fig. 39a is a girder with vertical bars supposed

I. Loaded on all its upper joints 1, 3, 5, 7, 9 (figs. 39a, 39b, 39c).

Consider §₁

| | | | |
|---------------------------|--------|------------------|------------------|
| Force polygon, fig. 33 | P | $\frac{Q}{0, 2}$ | $\frac{R}{0, 1}$ |
| In this diagram 39a, 39c, | P_A | $\frac{Q}{0, 2}$ | $\frac{R}{0, 1}$ |
| | given. | required. | required. |

Consider §₂.

| | | | | |
|-----------------|--------|------------------|------------------|-----------|
| Forces, fig. 33 | P | $\frac{R}{0, 1}$ | $\frac{Q}{2, 4}$ | S |
| In this diagram | P_A | $\frac{R}{0, 1}$ | $\frac{Q}{2, 4}$ | i. |
| | given. | given. | required. | required. |

From the extremity of $\frac{0, 1}{0, 1}$ draw i. vertically meeting the ray $\frac{2, 4}{2, 4}$.

Consider §₃.

| | | | | |
|-----------------|---------------------------|------------------|------------------|-----------|
| Forces | P | $\frac{Q}{2, 4}$ | $\frac{R}{1, 3}$ | S |
| In this diagram | $\frac{C(1, 3)}{C(1, 3)}$ | $\frac{Q}{2, 4}$ | $\frac{R}{1, 3}$ | b |
| | given. | given. | required. | required. |

Consider §₄.

| | | | | |
|--------|---------------------------|------------------|------------------|-----------|
| Forces | P | $\frac{R}{1, 3}$ | $\frac{Q}{4, 6}$ | S |
| | $\frac{C(3, 5)}{C(3, 5)}$ | $\frac{R}{1, 3}$ | $\frac{Q}{4, 6}$ | ii. |
| | given. | given. | required. | required. |

In this manner the diagram for a full load may be completed.

II. *Incomplete Force Diagrams for Partial Loading, fig. 39d.*—

In the cord polygon, fig. 39b, find the lines t' , t'' , t''' , and from them the points C' , C'' , C''' for loadings (41 and fig. 18), respectively over joints

3, 5, 7, 9
5, 7, 9
7, 9
9

i. For loadings over joints 3, 5, 7, 9.

Consider §₁. Focus C' .

| | | | |
|----------------------|--------|------------------|------------------|
| Force polygon | P | $\frac{Q}{0, 2}$ | $\frac{R}{0, 1}$ |
| In diagrams 39a, 39d | A' | $\frac{Q}{0, 2}$ | $\frac{R}{0, 1}$ |
| | given. | required. | required. |

Consider §₂. Focus C' .

| | | | | |
|---------------------------------------|--------|-------------------|-------------------|-----------|
| Force polygon | P | R | Q | S |
| In diagrams 39 <i>a</i> , 39 <i>d</i> | A' | $\overline{0, 1}$ | $\overline{2, 4}$ | i. |
| | given. | given. | required. | required. |

From the extremity of $\overline{0, 1}$ draw i. meeting $\overline{2, 4}$.

Consider §₃. Focus C' .

| | | | | | |
|---------------------------------------|--------|-------------------|--------------------|--------------------|--------------------|
| Force polygon | P | Q | R | S | R' |
| In diagrams 39 <i>a</i> , 39 <i>d</i> | A' | $\overline{2, 4}$ | $\overline{1, 3}$ | b | a' |
| | given. | given. | req ^d . | req ^d . | req ^d . |

Through point (1, 3) draw $\overline{1, 3}$, and from the end point of $\overline{2, 4}$ draw b .

ii. For loadings over joints 5, 7, 9.

Consider §₃. Focus C'' .

| | | | | |
|---------------------------------------|--------|-----------|-------------------|--|
| Force polygon | P | Q | R | |
| In diagrams 39 <i>a</i> , 39 <i>d</i> | A'' | (a') | $\overline{1, 3}$ | |
| | given. | required. | required. | |

Complete by triangle of Forces.

Consider §₄. Focus C'' .

| | | | | |
|---------------------------------------|-------|-------------------|-------------------|-----|
| Force polygon | P | R | Q | S |
| In diagrams 39 <i>a</i> , 39 <i>d</i> | A'' | $\overline{1, 3}$ | $\overline{4, 6}$ | ii. |

Through the end point of $\overline{1, 3}$ or (a') draw ii. meeting the ray $\overline{4, 6}$.

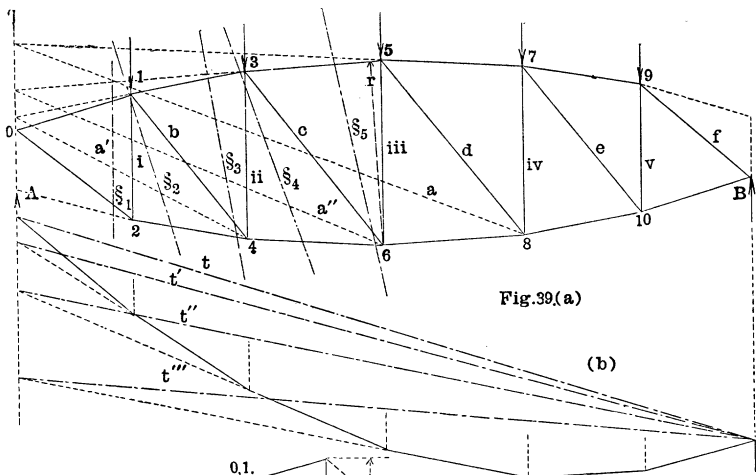
Consider §₅. Focus C'' .

| | | | | | |
|---------------------------------------|-------|-------------------|--------------------|--------------------|--------------------|
| Force polygon | P | Q | R | S | R' |
| In diagrams 39 <i>a</i> , 39 <i>d</i> | A'' | $\overline{4, 6}$ | $\overline{3, 5}$ | c | a'' |
| | given | given | req ^d . | req ^d . | req ^d . |

Through the point (3, 5) draw $\overline{3, 5}$, and from the extremity of $\overline{4, 6}$ draw iii.

In like manner we may obtain the stresses upon iii. and d for a loading on joints 7 and 9, and on iv. and e for a load on 9.

We have not complicated this diagram by taking into account the structural weight of the girder, but the amount of modification of the stresses which this occasions would be most easily obtained by the formation of a diagram of stresses for the structural weight alone, which could be added to those found by the partial diagrams, taking of course the sign of the stress into account.

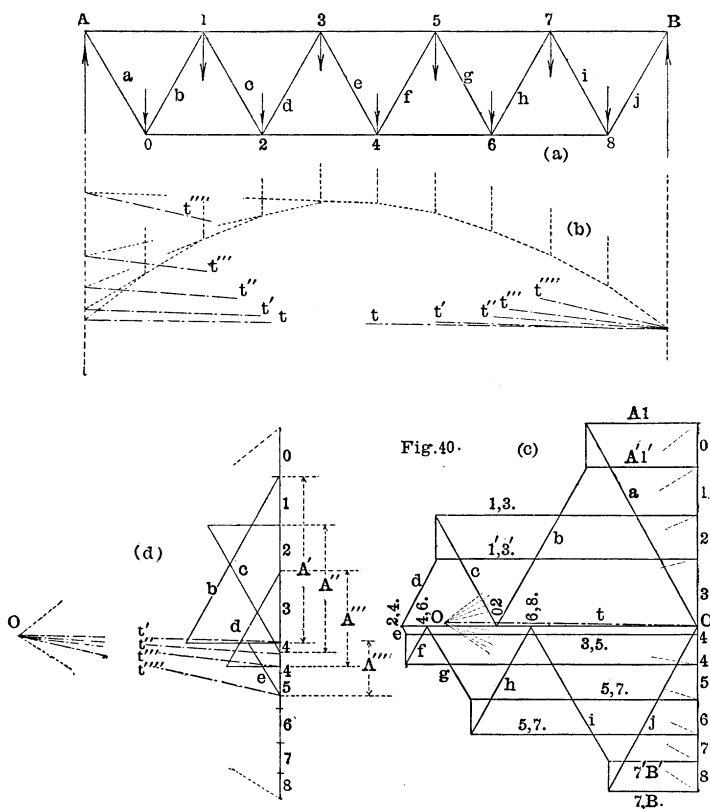


65. *Graphical Solution of the Analytical Expression for the Stress in the Bars of a Frame.*—Let it be required to find the stress on the bar 3, 5 under a full load, where r is the lever arm around the joint 6 (figs. 39a and 39e).

Transfer r to the ray e of the force polygon laying it off from pole O , from O likewise lay off on ray e the bending moment ordinate $M : h$ under joint 6. Then from the end point of r draw a line meeting the point H where the horizontal line h meets the line of weights, and parallel to this line draw a line through the end point of $M : h$, then

$$r : h :: \frac{M}{h} : \frac{M}{r}$$

: stress on 3, 5.



Section III.—Simplifications in Force Diagrams.

66. *Enumeration of Simplifications occurring in Force Diagrams.*—In forming force diagrams by the preceding method, certain dispositions of framing and forces lead to extensive simplifications.

i. There may be a simplification arising from a symmetrical disposition of framing and weights, the reactions at the points of support become equal, the line of weights is then bisected in the point C , which is therefore given, and the construction of force and cord polygons in order to obtain this point is dispensed with.

ii. Another simplification is frequently co-introduced, viz. considering the joints of the lower bars as unloaded.

iii. Another simplification occurs when the upper and lower bars form respectively a right line, as in the parallel boomed girder or form; each series respectively two straight lines meeting in the axis of symmetry is generally the case in roof framing.

In these cases the group of upper and of lower bars form each one and two groups respectively of parallel lines on the force diagram, the group or groups of lower bars proceeding from the focus C becoming superimposed.

These various simplifications in the force diagrams go a long way to conceal the steps by which they are obtained.

67. Force Diagrams of a Parallel Boomed Girder.

i. Under an arbitrary system of weights hung from each joint 0, 1, 2, 3 fig. 40a.

For this case the force and cord polygons (figs. 40b, 40c) must be drawn in order to find the point C .

The bars $\overline{A1}$, $\overline{1, 3}$, $\overline{3, 5}$ in the upper boom become parallel lines in the force diagram. The bars $\overline{0, 2}$, $\overline{2, 4}$ of the lower boom proceeding from C are superimposed upon each other.

The polygons in the force diagram are reciprocal, for instance in node 1, the rays in the node and the closed polygon sides in the force diagram are 1, $\overline{A'1}$, b , c , $\overline{1, 3}$.

GRAPHICAL DETERMINATION OF FORCES

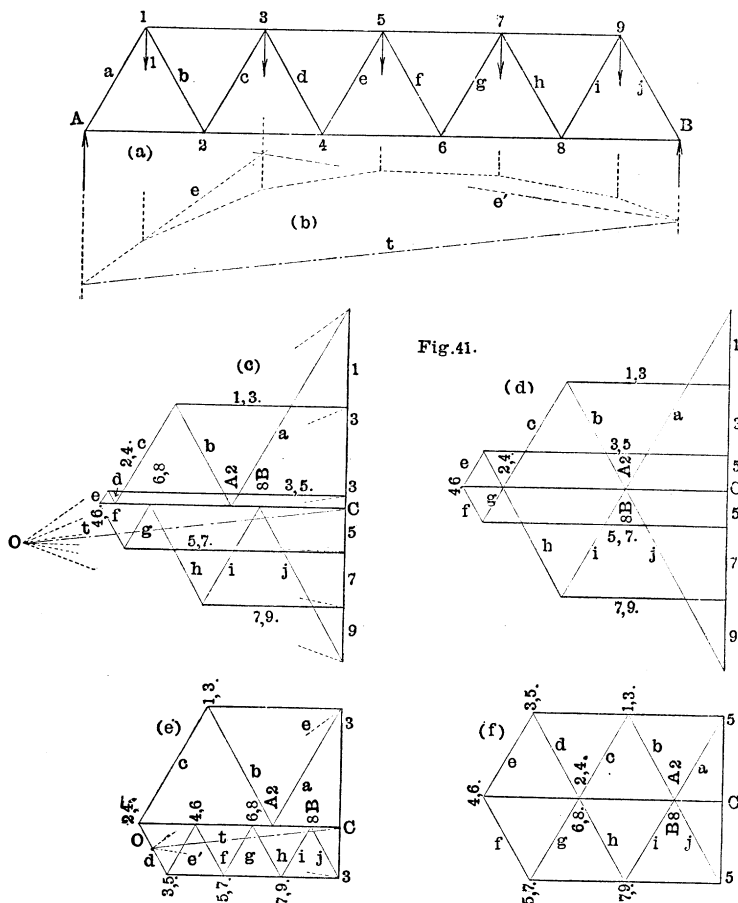


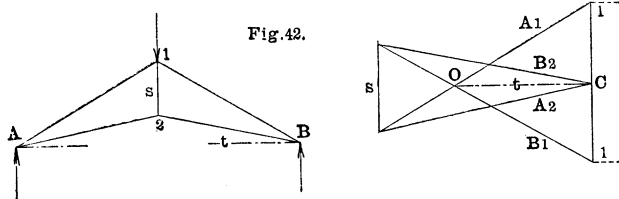
Fig. 41.

- ii. Under an arbitrary system of weights hung from the upper joints only (figs. 41a, b, c).
- iii. Loaded symmetrically on its upper boom (figs. 41a, d).
- iv. Loaded on one joint as 3 (figs. 41a, cord triangle e, e', t, in figs. 41b and 41e).
- v. Loaded on the central joint (figs. 41a, f).

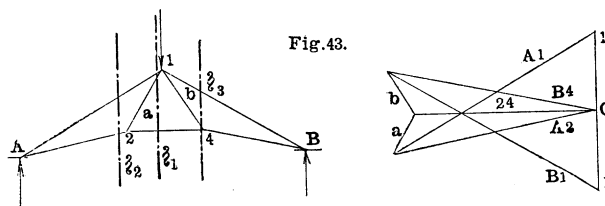
68. *Partial Force Diagrams in a Parallel Boomed Girder* (fig. 40d). This, after 64 II., requires no explanation.

69. *Force Diagrams of Roof Framing.*—As the simplifications of force diagrams, so apt to perplex, mostly occur in roof framing, we give a considerable number of examples.

I. Roofs with Simple Nodes.



i. *Simple Roof Framing.*—Fig. 42. This has analogy to fig. 30. In fig. 30 the disposition of the additional bars $\overline{A2}$, $\overline{B2}$ was to diminish the stresses in $\overline{A1}$, $\overline{B1}$. In this figure their effect is to increase these stresses.



ii. *Early Railway Roof.*—Fig. 43. Here the bars a , b , and $\overline{2, 4}$ take the place of the bar s of fig. 42. Beginning at A

$\overline{S_2}$ gives the stresses on $\overline{A1}$, $\overline{A2}$

$\overline{S_1}$ „ „ „ $\overline{2, 4}$, a

$\overline{S_3}$ „ „ „ b , $\overline{B1}$

$\overline{B4}$ equilibrates b and $\overline{B1}$.

iii. *Mansard Roof.*—Fig. 44. At the risk of prolixity we will go over the determination of the stresses in this roof in detail.

$\overline{S_1}$ gives $\overline{A1}$ and $\overline{A2}$, whence the triangle A , $\overline{A1}$, $\overline{A2}$.

Consider $\overline{S_2}$.

Force polygon

Fig. 44

$P_A - 1$

$\overline{C(1, 3)}$

given.

Q

$\overline{A2}$

given.

R

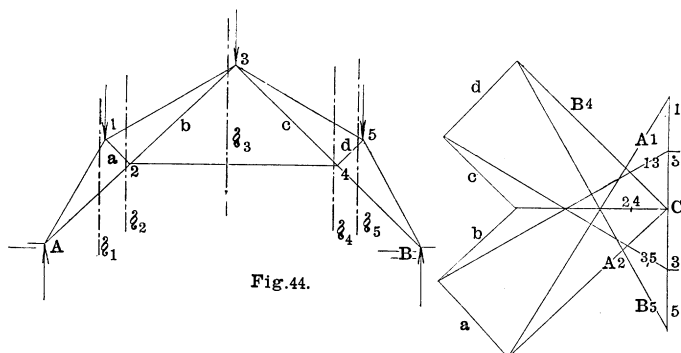
$\overline{1, 3}$

required.

S

a

required.



II. Roofs with Polyangular Nodes.

iv. *Swiss Roof*.—Fig. 45 is a simple example of this complex node.

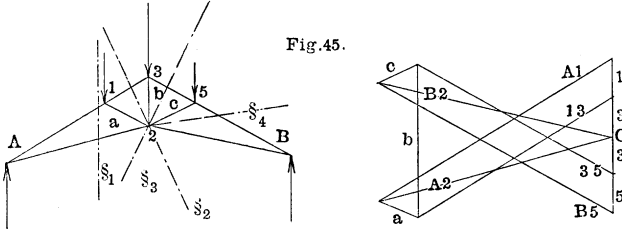


Fig.45.

Consider \S_1 .

From \S_1 we obtain the stress of $\overline{A1}$, $\overline{A2}$.

Consider \S_2 .

Force polygon

Fig. 45

$$P_A - 1$$

$$C, (13)$$

$$\text{given.}$$

$$Q$$

$$\overline{A, 2}$$

$$\text{given.}$$

$$R$$

$$\overline{1, 3}$$

$$\text{required.}$$

$$S$$

$$a$$

$$\text{required.}$$

Through point $(1, 3)$ draw $\overline{1, 3}$, and through the end point of $\overline{A2}$ draw a meeting $\overline{1, 3}$.

Consider \S_3 .

Force polygon

Fig. 45

$$P_A - (1 + 3)$$

$$\overline{C, (3, 5)}$$

$$\text{given.}$$

$$Q$$

$$\text{Resultant of } \overline{A2} \text{ and } a$$

$$\text{given.}$$

$$R$$

$$\overline{3, 5}$$

$$\text{required.}$$

$$S$$

$$c$$

$$\text{required.}$$

Through point $(3, 5)$ draw $\overline{3, 5}$, and from the extremity of the resultant r of $\overline{A2}$ and a draw c meeting $\overline{3, 5}$. Complete by aid of \S_4 .

v. *German Roof*.—When the Swiss roof (fig. 45) is so modified that the bars a and c are in one piece fixed horizontally, it becomes the “German roof.”

vi. *Crescent Roof with Polyangular Node at centre*.—Fig. 46. The weights have been taken irregularly, and the cord and force polygons from which the point C was found, have not been preserved.

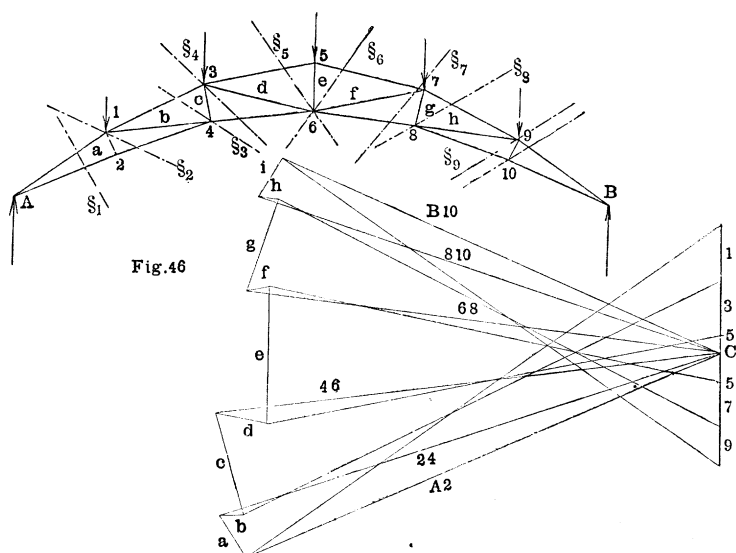
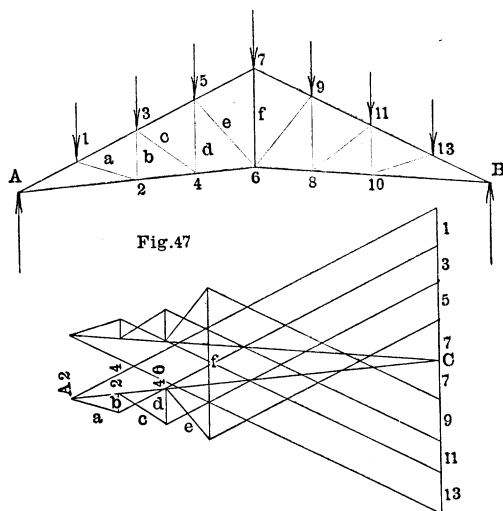


Fig. 46

| | | | | |
|---------------------------|----------------------|--------------------|--|-------|
| In § ₁ employ | P_A | $\overline{A1}$ | $\overline{A2}$ | |
| In § ₂ employ | P_A | R | Q | S , |
| that is | P_A | $A1$ | $\overline{2, 4}$ | a . |
| In § ₃ employ | $\overline{C(1, 3)}$ | $\overline{1, 3}$ | $\overline{2, 4}$ | b . |
| In § ₄ employ | $\overline{C(1, 3)}$ | $\overline{1, 3}$ | $\overline{4, 6}$ | c . |
| In § ₅ employ | $\overline{C(3, 5)}$ | $\overline{3, 5}$ | $\overline{4, 6}$ | d . |
| In § ₆ employ | $\overline{C(3, 5)}$ | $\overline{5, 7}$ | $(r = \text{result of } \overline{4, 6} \text{ and } d)$ e . | |
| In § ₇ employ | $\overline{C(5, 7)}$ | $\overline{5, 7}$ | $\overline{6, 8}$ | f . |
| In § ₈ employ | $\overline{C(7, 9)}$ | $\overline{7, 9}$ | $\overline{6, 8}$ | g . |
| In § ₉ employ | $\overline{C(7, 9)}$ | $\overline{7, 9}$ | $\overline{8, 10}$ | h . |
| In § ₁₀ employ | P_B | $\overline{9, 10}$ | $\overline{8, 10}$ | i . |

Our object in these two last examples is to familiarise the student with the treatment of a polyangular node. We have not transcribed on the force diagrams the names of all the bars.

vii. *English Roof*.—Fig. 47. This is but a simplification of the last example, and does not require therefore further elucidation.



III. *Trussing and Trussed Roofs.*

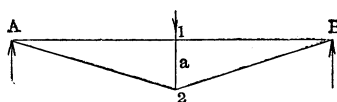
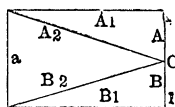


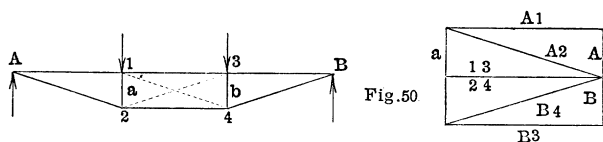
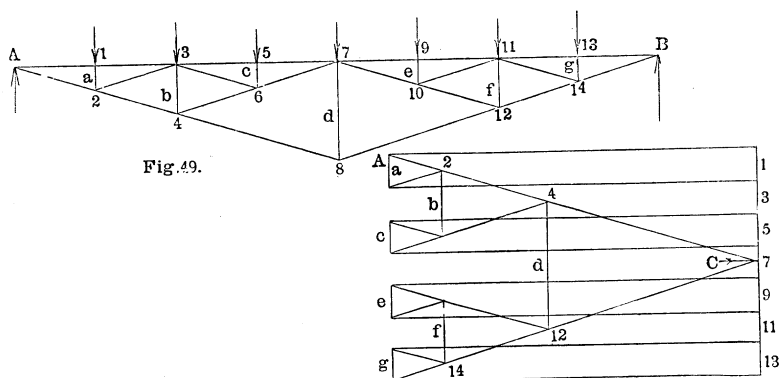
Fig. 48.



viii. *Triangular Truss*.—Fig. 48. This is the simplest form of trussing. It may be looked upon as a simplified form of fig. 30, in which the pole O is removed to an infinite distance, and on other grounds we also know that an infinitely great horizontal force can alone equilibrate a vertical force however small.

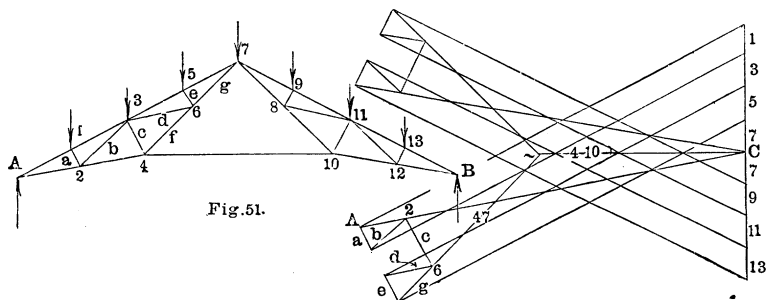
ix. *Compound Triangular Truss*.—Fig. 49. We give this truss and its force diagram without comment.

x. *Trapezoidal Truss*.—Fig. 50. This truss must be counter-braced for inequality of loading, for it is evident that then the shearing force has opposite signs at 1 and 3, and an unbalanced turning force exists.



xi. *French Roof*.—Fig. 51 is the frame diagram of the French roof, called by them “*Charpente à la Polonoise*.”

When the smaller trusses, that is, the bars a , b , d , e , are omitted, in large roofs, the member $\overline{A7}$ becomes a girder resting upon three points of support A , 3 , 7 , under the action of forces perpendicular to its direction and oblique forces along its length, and may require to be treated as a continuous beam.



Section IV.—CLERK-MAXWELL'S THEOREM.—Reciprocal Force Diagrams.

70. *Conditions necessary in order to obtain a completely Reciprocal Figure as Force Diagram to a given Frame divided into Triangles, are Equilibrium of Impressed Forces and their Cyclical Seizure in Force Diagram.*

In the method of sections (58) we have seen that we obtain for every node, taken separately, a reciprocal figure for force diagram. Attending now to fig. 54, and beginning at node *A*, we have the line $\overline{A1}$ in the force diagram, common to node *A* and adjoining node 1. We have also the force 1. Thus we have two lines of the reciprocal force diagram for node 1, which completed as that of *A* has been, gives the line $\overline{1,3}$ common to node 1 and adjoining node 3, and it gives no other line by which to step to another node, whence if we lay there at its extremity the force 3 we have again, two lines of the reciprocal force diagram of node 3. But the impressed forces being in equilibrium among themselves, form a closed polygon when seized in any order; regard being had to their signs, and hence also when seized in cyclical order. Now completing our reciprocal force diagram of node 3, we have so far a completely reciprocal figure, then seizing the next in order, and so on, we arrive at the lines $2A$ and a , already found, and which must again have the same values owing to the equilibrium of the impressed forces. Thus the whole figure is reciprocal.

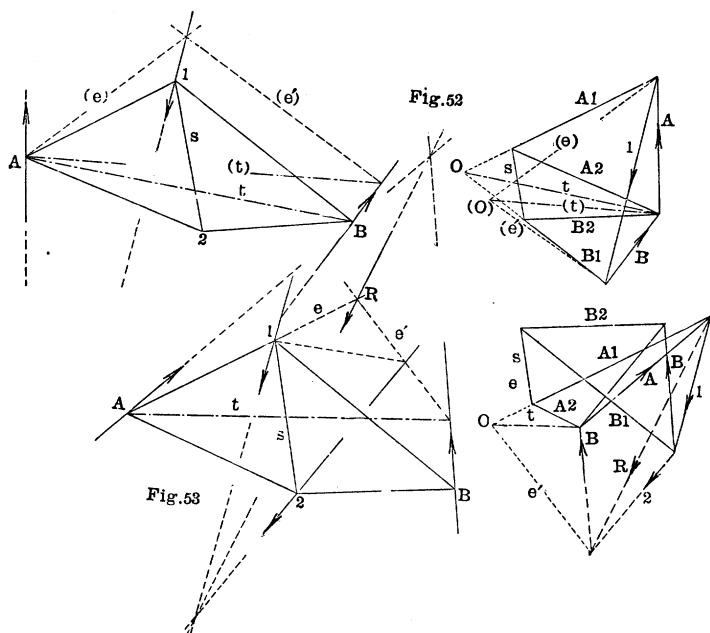
This may also be demonstrated after CREMONA¹ in a more general manner, but involving the consideration of forces in space.

It will be observed that the method of sections, by which we have till now obtained force diagrams, has led, in some cases, to partially reciprocal figures, *e.g.* figs. 37, 38 . . . and in others, to wholly reciprocal figures, *e.g.* figs. 46, 47 . . . and we can see that the failure to obtain figures wholly reciprocal occurs in those frames which are loaded on upper and lower nodes, for,

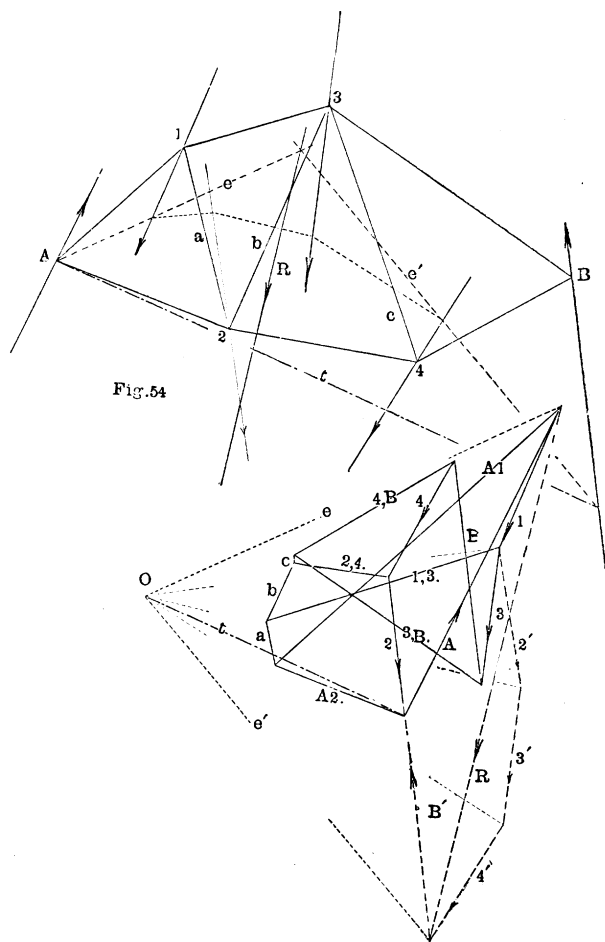
¹ Kräfte und Seilpolygon als reciproke Diagramme im Sinne Clerk-Maxwells.—*Zeitschrift des Oesterreichischen Ing.- und Arch.-Vereins*, 1873, p. 230.

when the upper nodes alone were loaded, the forces were seized cyclically.

In order to obtain equilibrium among the impressed forces of a beam under two supporting or reacting forces, we must, as in art. 61, fig. 36, assume the direction of one reaction, otherwise the problem is indeterminate.



71. *Examples of Formation of Reciprocal Diagrams to Frames of perfectly General Form with Forces in any Direction in a Plane.*—We give in figs. 52 and 53 two simple instances.—Fig. 52: Let 1 and the direction of B reaction be given, then with any point O as pole, we obtain the lines e, e' and consequently the line t of the cord polygon, which transferring to the force polygon gives the point C where B meets t whence A is determined. 1 is the resultant of A and B forming a triangle in the force polygon, and consequently forming a node in the lines of action of the forces produced (not shewn in figure). Beginning then with node A ; force A , bar $A1$, and bar $A2$ form a triangle in the force diagram passing on to force 1; force 1, bar $A1$, bar



s , bar $1B$ form a closed polygon; force B , bar $B1$, bar $\overline{B2}$ form a triangle.

Fig. 53: Form part of the force polygon with forces 1, 2, and the direction of the B force. Take any pole O , and by its means draw in the lines e , $\overline{1}$, $\overline{2}$, e' of the cord polygon, thus finding the lines t of force and cord polygons, and consequently the values and remaining direction of the reactions B and A .

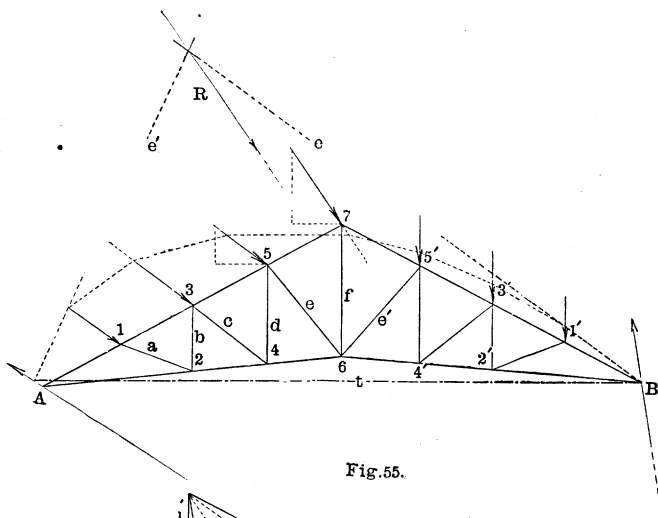
Beginning now with A , we have a node, force A , bars $\overline{A1}$, $\overline{A2}$ with which to form a triangle in the force diagram. Pass on to

node 1, then force 1, bars $\overline{1A}$, s , $\overline{1B}$ form a closed quadrilateral. Pass on to node B in the force polygon, transfer it from the end point of force 2 to the end point of force 1, and force B , bars $\overline{B1}$, $\overline{B2}$ form there a triangle. Now pass on to 2; force 2, bars $\overline{2B}$, s , and $2A$ form a triangle, and the reciprocal figure is complete.

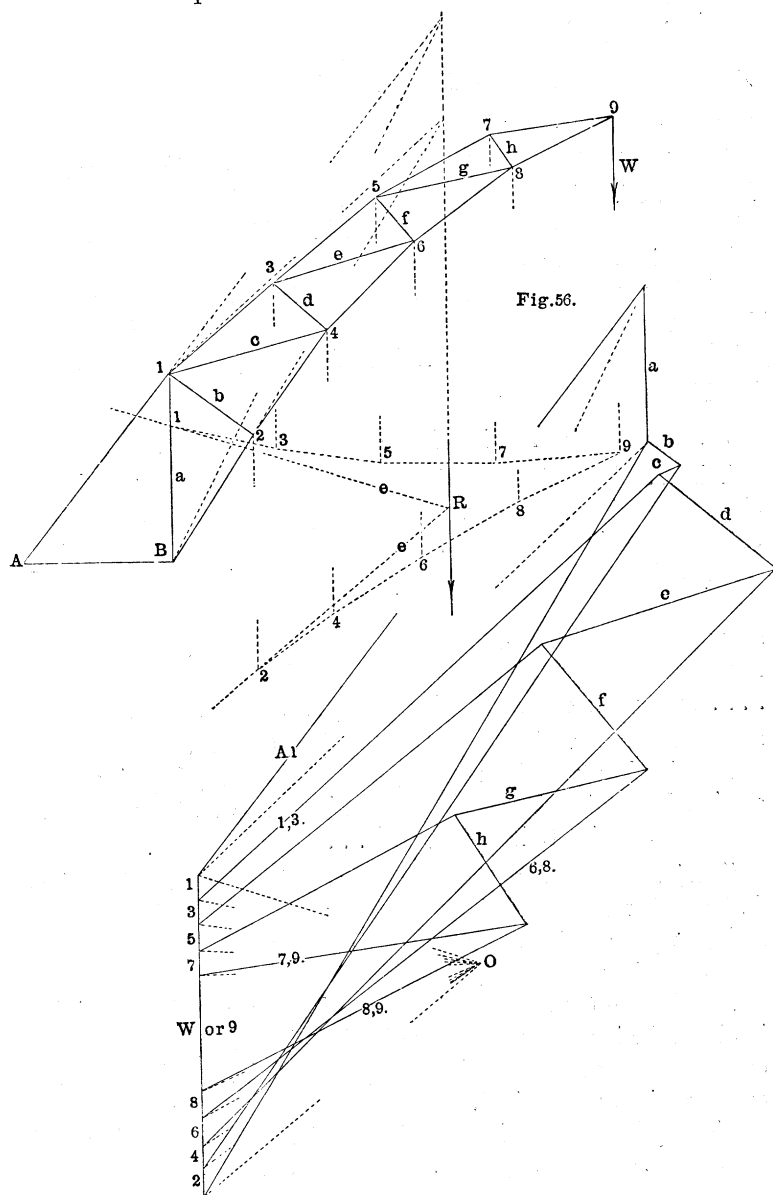
Fig. 54 is more complex, but the operations are the same as in the last figure.

i. The ordinary line of weights of the known forces 1, 2, 3, 4, together with the direction of one of the reactions B is laid down.

ii. The cord polygon is now formed, giving the line t .



iii. The line t is transferred to the force polygon, by which means it is completed.



iv. The forces are seized in *cyclical* order beginning at *A*, then node 1, then node 3 which requires the force 3 to be transferred to the end point of 1, then node *B* which requires the reacting force *B* also to be transferred, then node 4, then node 2; forces *A*, *B*, and *R* in frame diagram unite in a node beyond the boundary of the page.

72. *Application of Reciprocal Diagrams to Wind-pressure on a Roof, fig. 55.*—Although this can be accomplished by the method of sections, yet carrying the idea of a reciprocal figure in the mind facilitates the construction.

Having compounded fig. 55, the wind and roof pressures 1, 3, 5, 7, and equilibrated the impressed forces, by giving to one of the reactions as *A* a determined direction, we proceed as in the previous examples.

73. *Examples of Application of Preceding Methods.*

i. Fig. 56 is the framework of a *Crane* after CREMONA, and may either be looked upon as worked out by the method of sections or that of reciprocal figures.

In order to its verification the line of action of resultant *R* was found by means of the ordinary force and cord polygon, shown in dotted lines. The last three bars *A1*, *B2*, *b*, of the figure are the *R*, *Q*, and *S* of the general solution, figs. 32, 33 . . . and with either the imaginary links *Q'* and *R'* of that solution we can verify the points (*b*, *a*) or ($\overline{A1}$, *a*).

The load, it will be noticed, is not constant, but increases section by section, being, first, at node 9, $P = W$, for section to the right of joint 8, $P = (W + w_s)$. . .

ii. Fig. 57*a*, 57*b*, are frame and force diagrams of a *Turning-Bridge* after LEVY,¹ and, like the previous example, may be looked upon as either worked out by the method of sections or that of reciprocal figures.

In this case (1) the line of the resultant *R* of all the weights is given, for it must pass through the pivot *A* (2) the weights over the span are supposed known, and (3) the weights over the counterpoise are required.

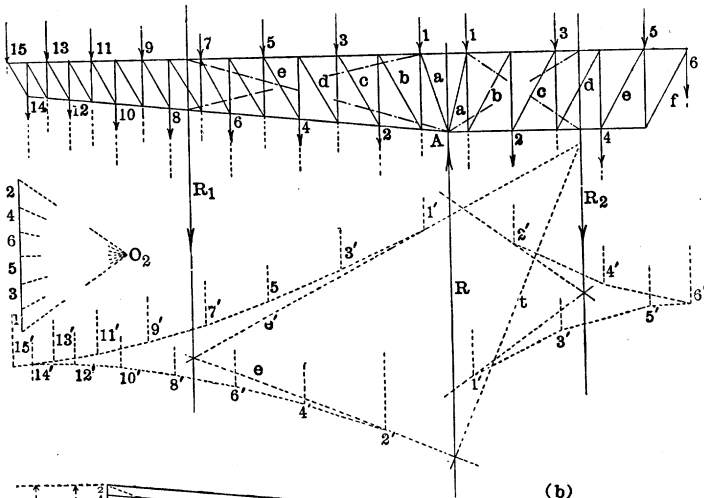
Finding by means of force polygon O_1 , and corresponding cord

¹ *La Statique Graphique*, Maurice Levy; Paris, Gauthier-Villars.

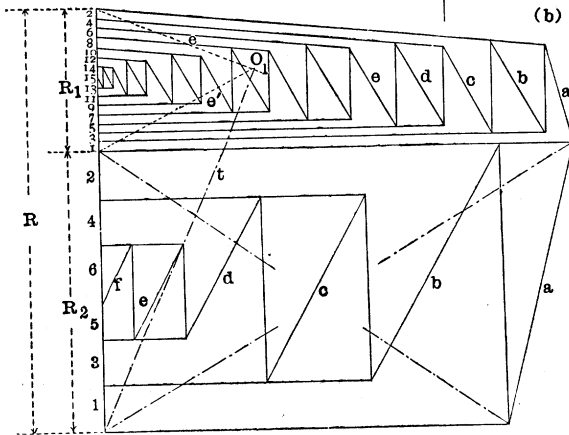
polygon the line of action of resultant R_1 of the weights over the span, then, supposing the weights over the counterpoise distributed in any given ratio, say, equally over the joints, form a force polygon O_2 with its line of weights equally divided, and by means of its corresponding cord polygon, find the line of action of the resultant R_2 when we have a case of art. 20. Given the lines of action of three forces R, R_1, R_2 in equilibrium, and the value of one of them R_1 , to find the values of the other two.

Fig. 57.

(a)



(b)



Solved as follows, extending the extreme rays in the cord polygon of O_1 , till they intersect respectively the line of action of R_2 and of R , we obtain the closing line t of O_1 and the values of R and R_2 as indicated.

Divide R_2 into the number of equal parts corresponding to the number of compartments in the counterpoise, and proceed to find the stresses as in last example.

The accuracy of the stress diagrams may be tested as formerly by the reciprocals in the force diagram of the imaginary links Q', R' , in the frame diagram, shewn dotted and partly drawn in, in the figures.

iii. Fig. 58 is an example of the reciprocal force diagram of a *Hanging Girder* loaded on both upper and lower joints.

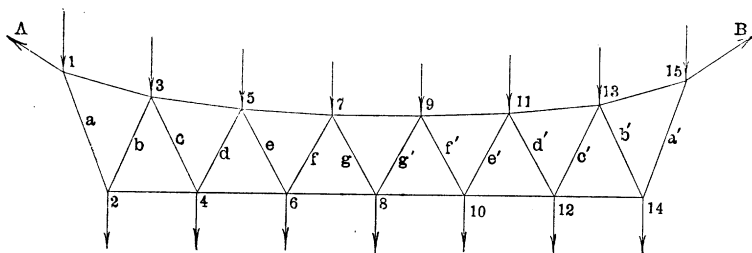
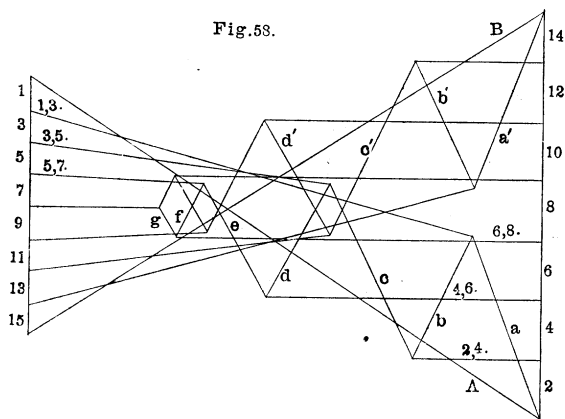


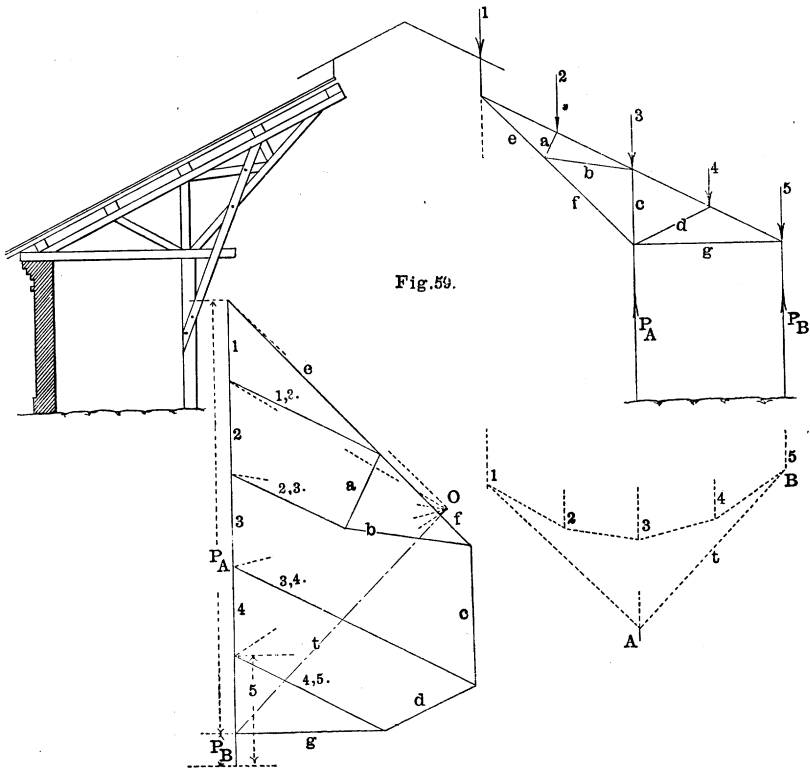
Fig. 58.



iv. Fig. 59 is an example from CREMONA of an *irregular roof frame* and its reciprocal force diagram.

Section V.—Herr HEUSER'S¹ Problem and its Application to Hinged Arch Work.

74.—*Problem. To Carry a Cord Polygon to given Forces through Three given Points.*—Let (fig. 60) 1, 2, 3, 4 . . . 9, be the lines of action of a number of forces and *A*, *B*, *C* three given points, we



can then, with any pole distance in the force polygon, describe a cord polygon going through the given point *A*; or, for greater clearness of figure, let us first describe a cord polygon upon the lines of action of the forces through no particular points, fig. 60*a*, with any pole distance *O'*. This gives us the line of action of

¹ "Zur Stabilitätsuntersuchung der Gewölbe," *Deutsche Bauzeitung*, 1872, p. 365.

the resultant R_1^9 of all the forces. Let the point C , through which we wish to carry the ultimate cord polygon, lie between forces 6 and 7; then, by extending the link $\overline{6, 7}$ till it meets the extreme rays e' and e'' we can, with the help of the force polygon, obtain the lines of action of the resultant forces R_7^9 and R_1^6 .

Through the points A and B draw two lines e'' and e' intersecting in the complete resultant R_1^9 ; e'' and e' are necessarily the two extreme rays of a cord polygon of forces 1, 2, . . . 9 going through A and B .

Through the intersection of the extreme rays e' , e'' with the partial resultants R_7^9 and R_1^6 , draw a line f'' meeting the line through AB in F , then f'' is the line $\overline{6, 7}$ extended of the cord polygon going through A and B whose force polygon pole is O'' , but which cord polygon it is unnecessary to complete.

Again, as A, B on the line AB are the points through which the extreme rays e, e' of the ultimate cord polygon must pass, we have two homologous sides of two cord polygons cutting on the line AB , e and e'' on the right, pivot around B , e and e' on the left, pivot around A , whence all other homologous sides must pivot around points on this line, whence the sides $\overline{6, 7}$ of all cord polygons to these forces going through A and B must pivot round F ; whence the side $\overline{6, 7}$ of the ultimate cord polygon must pass through F (52 and fig. 29b); whence

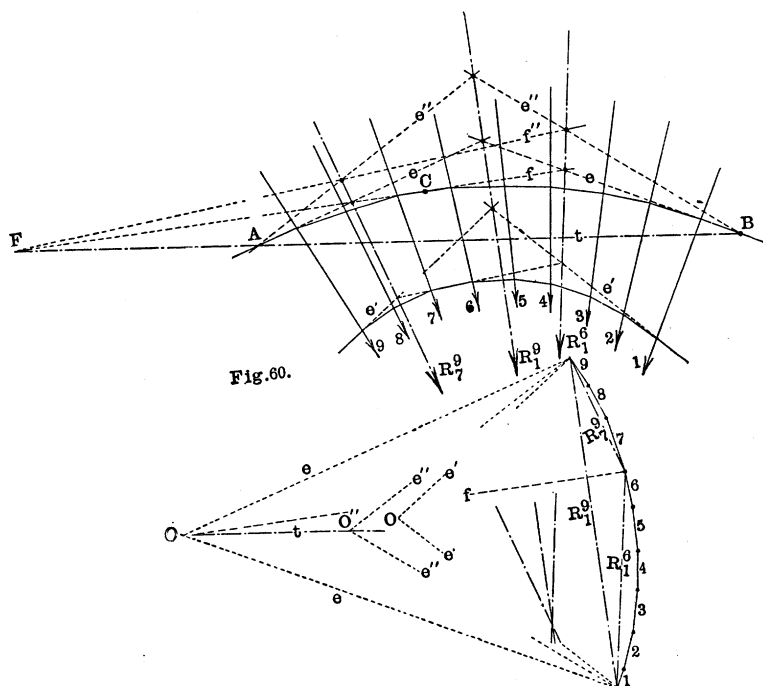
Through the points F and C draw a line f which will be cut by the lines of action of the partial resultants R_7^9, R_1^6 in two points. Through A and B and these two points draw two lines e and e' , these lines will intersect the complete resultant R_1^9 , and the parallels to them in the force polygon will intersect in the point O on the line t through O'' parallel to t . This point O is the pole of the force polygon for the ultimate cord polygon. (Compare fig. 29b.)

This proposition is manifest from more circumstantial reasoning: t, e'', f'', e' , is a cord polygon of two forces R_7^9 and R_1^6 and of their reactions at A and B , F is a point in the line of action of the force $(A - R_7^9)$; t, e, f, e' is a cord polygon of the same forces, whence the side f necessarily passes through the same point F .

When F is inconveniently distant, lines proceeding towards it may be obtained by means of a problem given in Chap. IX.

75. *Application of HEUSER'S Problem to Hinged Archwork.*—Fig. 61 is an example of hinged archwork, being hinged at the crown C and springings A and B , supposed loaded as far as node $3'$ with structural and travelling load, from $5'$ to $11'$ with structural load only.

The constructions of the preceding problem (Heuser's) can easily be recognised in the figures. By its means we obtain the line of stress Q through C and its value in the force polygon.



Then BC and AC are treated as two distinct frames whose supporting forces are respectively S and Q and S' and Q' . The other impressed forces are respectively 1, 3, 5, 7, 9, 11 and $1'$, $3'$, $5'$, $7'$, $9'$, $11'$.

The stresses in the force diagrams can now be resolved by the

preceding methods of the chapter of which the method of reciprocal figures is the most convenient.

It is scarcely necessary to remark that, in this example, AC and BC being each in a straight line the stresses in $C2, 2, 4, \dots$ and $C2', 2'4' \dots$ are all measured from O but are superimposed on each other.

In an example in which AC and BC were curved, the parts $\overline{C2}, 2, 4, 4, 6, \dots$ would be taken as straight lines, and we would have a pencil of rays $\overline{O2}, \overline{O4}, \overline{O6}, \dots$ at O measuring their stresses respectively.

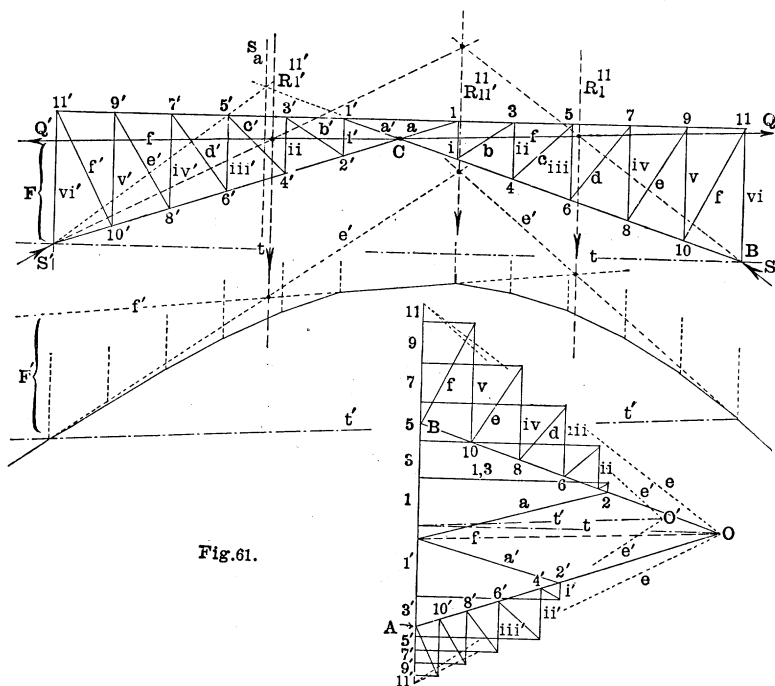


Fig. 61.

Section VI.—Compression and Tension in Bars.

76. *Analysis of the Conditions giving Compression or Tension in Bars.*

- i. A force acting upon the joint of two bars and *between* the

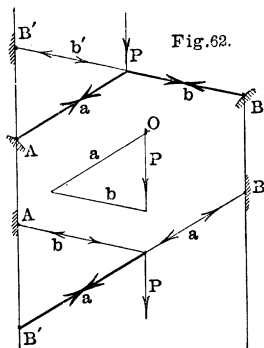
two supports of these bars. The bars are in ^{compression}_{tension} according as they are ^{convex}_{concave} to the direction in which the force acts (fig. 62).

ii. But a force acting upon the joints of two bars whose two supports are on one side of the force—

A bar is in ^{tension}_{compression} according as that bar produced gives a form ^{convex}_{concave} to the direction in which the force acts.

77. Rule by which to Discover the Sign of Stress in a Bar.—

If then, in the force diagram of two bars with a force on the node, it appears that, causing the pencil to travel over the stress lines of the bars from the origin O of the force till it arrives at the other end of it, and then, placing the pencil at the joint P , causing it to travel along any bar or the continuation of any bar in the direction travelled over previously on the stress line of the same bar in the force polygon, then the bar is in (1) *compression* or (2) *tension*, according as the pencil moving from the joint has travelled over the (1) *bar* or the (2) *continuation of the bar*.



What is true of a polygon of three sides is true of one of many, and the enunciation of the rule is the same.

We will not formally demonstrate the statements in this and the previous paragraph.

Section VII.—Limit to which a Travelling Load must extend over an Open Framework in order to give a Maximum Stress in any given Bar.

78. *Limit for a Girder of General Form, fig. 63.*—In this figure, we observe that if a weight W be placed over a joint B , its whole weight is employed in adding to the stress on that

joint, and if this weight be moved forward (in our figure to the left), its effect on joint B diminishes and its effect on joint A increases, but at first, its influence on joint B is greater than its influence on joint A until a point W is reached where its influence on joint A is equal to its influence on joint B , and beyond which its influence on B would be less than on A .

It follows from this, that between B and the point W , a series of loads such as wheels may be placed, each of which adds to the stress on the bar b , in proportion to its weight, combined with its proximity to B , wherefore (40, II.) the strain on the bar b for any given series of loads is a maximum, when the bridge is loaded up to the point W .

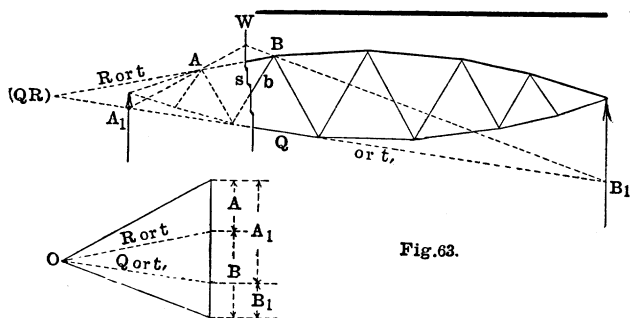


Fig. 63.

Fig. 63 shows how this point is obtained. Extend Q through verticals of support, giving the points A_1 , B_1 . Join A_1A , B_1B , produce them till they intersect in W , then the vertical through W cuts the platform of the bridge in the point up to which it may be loaded in order that the bar b may be subjected to a maximum stress.

For, consider W as the resultant of two loads at A and B respectively, whose reactions are A_1 , B_1 , then A_1ABB_1 may be considered a cord polygon (Q, R) BB_1 a cord polygon for the weight at B (for QR is the intersection of t and AB , and consequently (42) a point in the line of reaction of $B - B_1$). As then $B - B_1$ passes through QR it exerts no influence on b , and as

$$A_1 + B_1 = - (A + B)$$

$$A_1 - A = - (B - B_1),$$

then in substituting the resultant of $B_1 - B$ at (Q, R) we do away with its equivalent $A_1 - A$.

Now consider a weight to the left of W , it is evident that the resultant will pass through a vertical to the left of W and influence b negatively, on the other hand, all weights to the right of W influence b positively, W therefore is the furthest point on the bridge up to which a load must be brought in order to increase the stress on b .

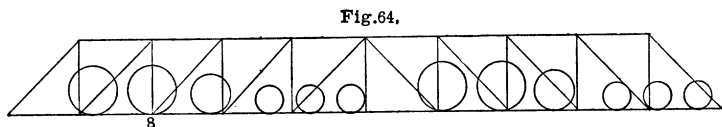
In the above, we have supposed the pressures to the right of B not to be altered, but this is not strictly the case, for in moving forward a travelling load, we move forward the resultant of the whole and may thus move the point C in the force polygon upwards (figs. 39c, d) to more than neutralise the effect of the advance of the load beyond B .

It further requires this modification that the maximum stress on a bar b must occur when a wheel lies over a joint B , so that we must limit the rolling forward beyond B to an integral number of wheels.

79. *Limit in a Parallel Boomed Girder.*—In a parallel boomed girder where this construction fails the formula is simple. Let ΔP be the weight of the wheels rolled over between B and W . Let n be the number of compartments in the girder (supposed equal), then for a maximum stress on b

$$n \cdot \Delta P = P.$$

Consider such a girder for instance as fig. 64, having $n = 10$,



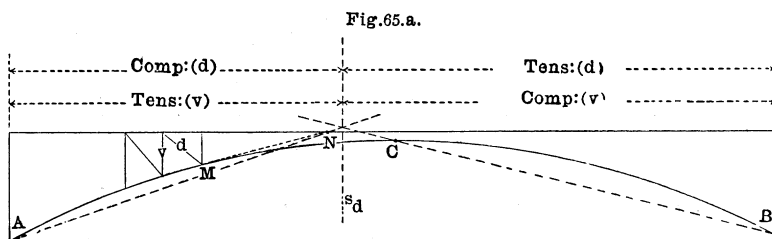
with a train of engines and tenders upon it, then (fig. 7) taking the weight of the first wheel within the points of the compasses and stepping down the line of weights, we find that $9 \times$ weight of first wheel comes between the weight of the eleventh and twelfth, so that when there are more than eleven and less than twenty-two wheels upon the bridge one wheel must be rolled over the joint, fig. 64, where this occurs at the eighth joint

whence a wheel must be rolled over for the eighth and ninth joints; when there are less than eleven wheels, no wheel must be rolled over.

80. *Limits in a Hinged Archwork.*

i. *Limit for Diagonals, fig. 65a.*—Let the load come on from B proceeding to the left.

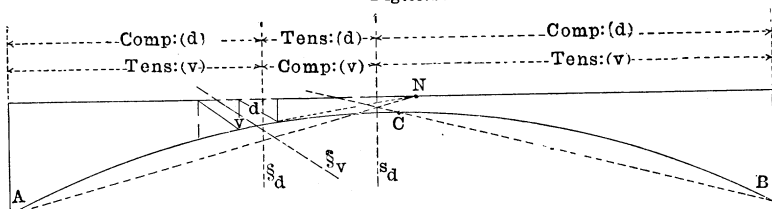
Through B and C draw a line, from the point M where the diagonal meets the arch draw a tangent MN meeting the extrados in N . Through A and N draw a line intersecting BC . Through the point of intersection draw the vertical s_d , where s_d cuts the platform is the furthest limit for a load giving a maximum on the diagonal, d .



The point N in this figure is the point (Q, R) of fig. 63. The extrados forming the Q and MN the R of fig. 63 and of figs. 32 and 33, whence a force through N exerts no influence on diagonal d . Now a weight whose line of action is in s_d by the conditions of the frame acts in CB and AN , whence AN is the line of action of the weight in s_d and exerts no influence on diagonal d . All forces to the right of s_d have their resultant with BC , passing in a line through A and through a point to the right of N , thus affecting diagonal d with stresses all of one sign. On the other hand all forces to the left of s_d have their resultant passing in a line through A and through a point to the left of N thus affecting diagonal d with stresses all of the other sign, s_d being the limit in which the influence of a weight passes through zero (78).

We have a maximum of $\begin{matrix} \text{tension} \\ \text{compression} \end{matrix}$ according as the bridge to the $\begin{matrix} \text{right} \\ \text{left} \end{matrix}$ of s_d is loaded and the rest of the bridge unloaded.

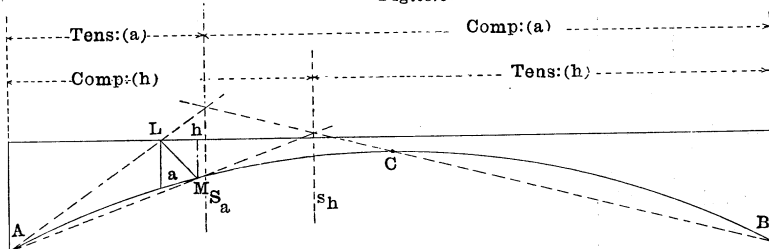
Fig. 65.b.



ii. *Limit for Diagonals near Summit of Arch, fig. 65b.*—In diagonals near summit of arch, the point N lies on the other side of the line BC , and then a section S_d through the diagonal d is a second limiting line, so that we have a maximum of tension arising from full load between S_d and s_d the rest of compression no load the bridge being unloaded loaded.

iii. *Limit for Verticals, figs. 65a, 65b.*—The limit for verticals v is s_d as for diagonals, and in the case of verticals near the summit of the arch there is also a second limiting line S_v , viz. that through the vertical under consideration, fig. 65b, the conditions of loading that give compression to the diagonals giving tension to the diagonals giving tension to the verticals. compression

Fig. 65.c.



iv. *Limit for Extrados, fig. 65c.*—Let the load as formerly come on the frame from B proceeding to the left.

Through B and C draw a line. Through the points A and M draw AM cutting BC and through the point of intersection draw the vertical s_h , where s_h cuts the platform is the limit for a load giving a maximum to the extrados h .

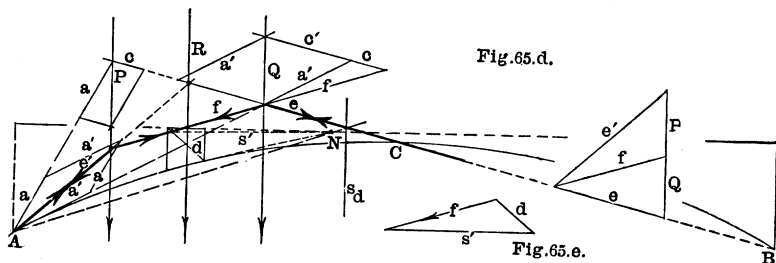
A weight, as formerly in the vertical of s_h , by the conditions of the frame acts in BC and AM , and thus exerts no influence on the point M . The point M is the turning point and is in fact for the extrados h the (Q, R) of figs. 32 and 33, LM and vertical are the Q and R of these figures, and according as the weight is to the left or to the right of s_h , so does the force through AW act above or below the point M , and affect the extrados h with opposite signs.

We have in fact a maximum of tension in extrados h according as ^{right}_{left} from s_h is fully loaded, the rest of the bridge being unloaded.

v. *Limit for Quasi Arch Member, fig. 65c.*—Draw the line AL meeting BC , and through the point of intersection draw the vertical s_a , where s_a cuts the platform is the limit for a load giving a maximum to the arch bar a .

The point L is here the turning point, whence the resultant AL is in that line of action where its influence on a passes through zero and changes sign, whence, &c. (78).

The considerations of this section are required in the formation of partial diagrams. Thus in fig. 61 the limit for a maximum on quasi arch member 4'6' has been taken, so far as the supposition of concentrated weight over the joints allows.



81. *Partial Diagrams of Maximum Stress in Hinged Archwork.*

i. *For Diagonals, figs. 65d, 65e.*—We may obtain the resultant force acting on diagonal by a modification of Heuser's problem. Find, by means of a cord polygon if necessary, the resultants P

and Q of all the forces acting on the left and right hand respectively of a section through the diagonal in question and their lines of action, also the resultant R of all the forces and its line of action, then carry a cord polygon of these two partial resultants with the help of the line of action of R through the two points A and C and having BC for one of its extreme rays e . We thus find the ray f of the cord polygon, and its value from the corresponding force polygon.

The ray f is the resultant force acting upon the diagonal d , and is the P of fig. 32. In fact comparing figs. 32*a*, 32*b*, with figs. 65*d*, 65*e*, P in the former is f in the latter.

The point (PS) in the former is the point (f, d) in the latter.

" " (QR) " " " " N " "

" line S' in both figures is the line $\overline{N(f, d)}$ " "

" " S in the former " " d " "

Whence the partial force diagram, fig. 65*e*, corresponding to fig. 32*c*.

The following considerations enable us to simplify the foregoing method of obtaining f . From the nature of the frame, P and Q necessarily decompose themselves into forces P_c, Q_c along the continuation of BC and into forces P_a, Q_a radiating to the point of reaction A , (the forces P_a, P_c are marked on fig. 65*d* by a, c , and the forces Q_a, Q_c by a', c'); and a little consideration of fig. 65*d* will show without formal demonstration that f may be regarded as the resultant of R_a and Q_c , that is, of Q_a, P_a and Q_c , or of Q_a and P_c , and in the example we will presently give, it will be regarded as the resultant of Q_a and P_c (a' and c of the figure).

The continuation of BC beyond C on the left member of the bridge, and the continuation of AC on the right member of the bridge is the locus of the intersection of the resultants $P_a P_c$ of any weight P upon it passing through the springing points A and B .

ii. *For Diagonals near Summit of Arch.*—In this case we only require the resultant of Q through A as the value of f . Fig. 65*f*.

iii. The reason for loading with both P and Q in the lower set of left-hand diagonals, having the point N on the same side

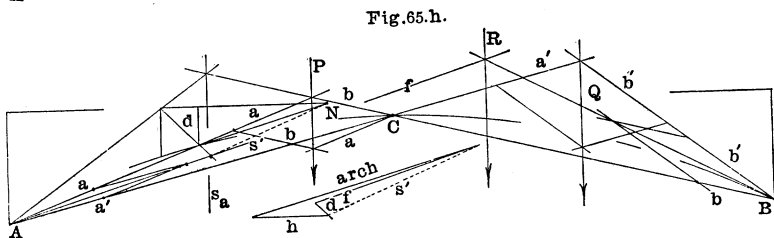
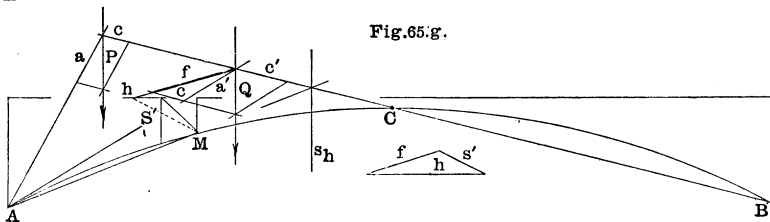
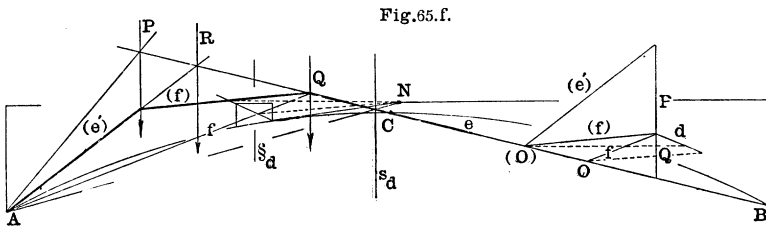
of BC with themselves, and loading only with Q in the upper left-hand set of diagonals having the point N on the right side of BC , will be apparent from the following considerations. If the point N were upon the line BC , we have upon the arch the quadrilateral formed by the diagonal d , the line BC , the diagonals f and (f) , of which the combined force polygon d , O , (O) , f , and (f) , is a reciprocal figure, and the value given to d would be the same in both cases (51). In fact \overline{QN} is the line in all cases which would have to be employed as the line $\overline{O(O)}$ in order to obtain a reciprocal force polygon (51), whence, when N is to the left of BC , (O) will be to the right of BC giving necessarily a less value to d but equal to that given by Q alone, whence P_c adds to the value of d in this case.

When again N is on the right side of BC , then (O) would require to be to the left of BC in order to give a reciprocal figure, and in this instance would give the same value to d as Q alone, whence with (O) on the line BC , the ray (S') is moved upwards parallel to itself, giving a less value to d than from Q alone.

iv. *Partial Diagram for Extrados h , fig. 65g.*—Here we have found f without constructing the cord polygon, by means of the simplified method of this article. M is now the point (Q, R) of fig. 32, and S' now joins the point M and the point (f, h) , h being S of fig. 32.

v. *Partial Diagram for Arch Member, fig. 65h.*—Here the limit lies between B and s_a . The load naturally divides itself into two loads P and Q , one on either side of the point C , which resolve themselves in the structure each into two other forces P_a and P_b and Q_a , Q_b ; P_a and Q_a going through the springing A , P_b and Q_b going through the springing B . P_a and P_b are marked a and b on drawing, Q_a and Q_b are marked a' and b' . These four forces are compounded in the figure into R_a and R_b by the parallelogram of forces, R_a being the force f acting upon the arch. It meets the diagonal d in the point (d, f) and, continuing the line of the arch member till it meets the extrados h in the point N , join (d, f) and N giving the link S' , then the force diagram is equivalent to fig. 32c, arch and h being the Q and R , d and S' the S and S' of that figure.

82. *Combined Series of Partial Diagrams giving Maximum Stresses in Hinged Arch Work.*—The methods above unfolded can be applied to the arch loaded in any manner, *e.g.* with a train of locomotives and tenders, the resultant weights of which can be obtained from a diagram such as figure 7, but on the supposition of uniform weights being placed over every vertical bar, we can form expeditiously a figure giving the maximum values of the stresses of the four series of bars, viz. *f* diagonals, verticals, extrados, and intrados.



In fig. 65*i*, form the auxiliary figure (2) thus:—

The vertical OO' measures the load over a joint. From O lead one pencil of rays $O1, O2, O3 \dots OC$ alternately similar to a pencil $A0, A1, A2$, from A , the points $1, 2, 3 \dots$ being vertically over the joints $1, 2, 3 \dots$ and upon the locus of the intersection of the resultants P_a, P_b (81, i.), and a second pencil of rays $O1, O2 \dots OC$ alternately similar to a similar pencil from B . From O' lead two transversals $O'C, O'C$, parallel

to BC , AC , then we see at once that any one of the rays as $O5$ is the P_a of a load OO' over joint 5, whilst $O'5$ is the P_b of the same load; P_a going through A , P_b going through C and B .

In figs. 3 and 4 lay off $O'1$, $O'2$, $O'3$ $O'C$, $O9$, $O8$, $O7$ $O1$, OO' .

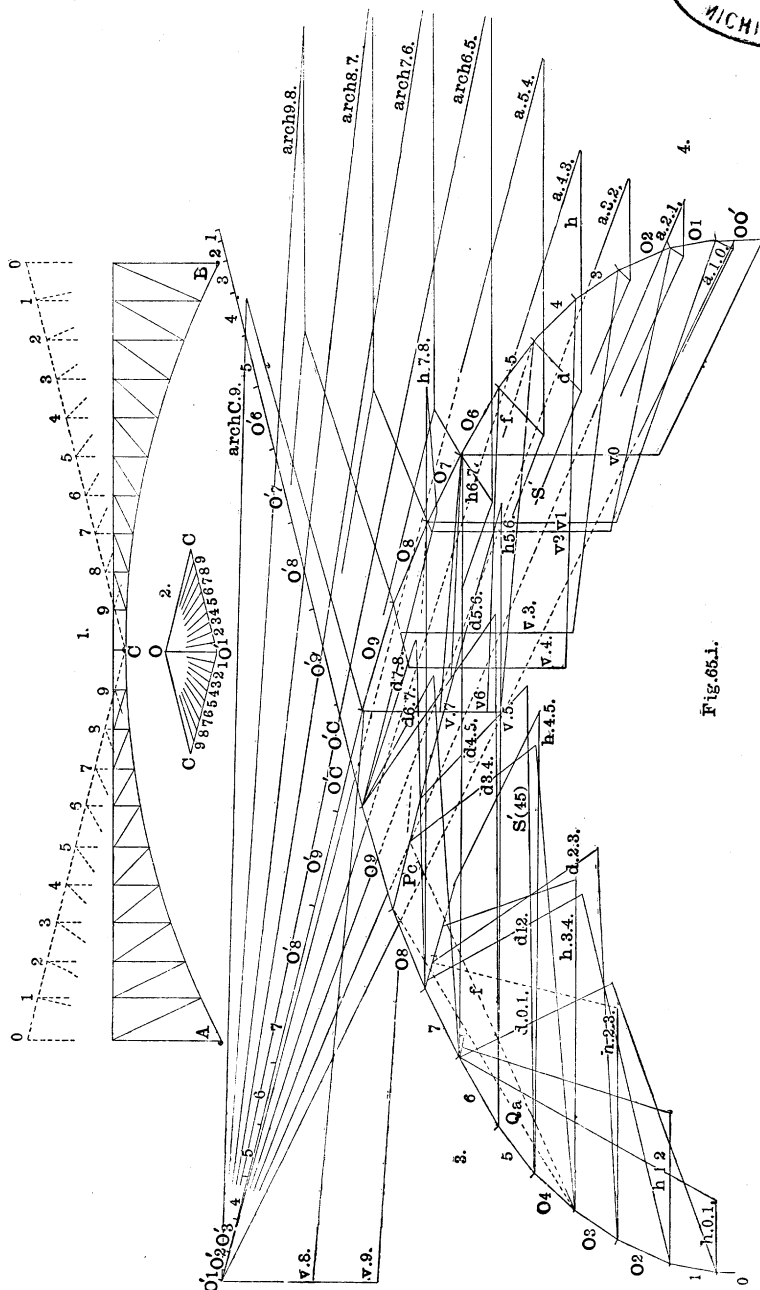
We are now in a position to form the four series of partial diagrams. Take for instance the compartment 3 to 4 on the left-hand side. Then for the diagonal 3, 4, the load P on the left is 1, 2, 3, the load Q on the right is 4, 5, 6, 7, 8, then a line joining the points $(O3, O4)$ and $(O8, O9)$ is Q_a (compare fig. 65*d*), and a line equal and parallel to $(O'1 + O'2 + O'3)$ of fig 4, and drawn from the end point of Q_a is P_c . These give f . From A draw a line (not shown in 65*i*, but compare 65*d*) parallel to Q_a of fig. 3, and from the point where it cuts BC continued draw in f ; we now obtain S' and can finish a partial diagram similar to 65*e*. Q_a , P_c , f , S' , are marked on fig. (65*i*, 3). S' has been marked, however, for 4, 5 instead of for 3, 4.

For the partial diagram for arch bar 4, 3, on the right side, fig. (65*i*, 4), f , d , S , h , are marked and only require comparison with fig. 65*h* for its full comprehension.

On fig. (65*i*, 3), are worked out the partial diagrams of the extrados and diagonals. On fig. (65*i*, 4), those of the arch and verticals.

To the results obtained in this manner must be added (having regard to sign) the stresses arising from structural load only, which are best obtained by means of a complete reciprocal force diagram after art. 75, and fig. 61.

83. *Corresponding Maxima and Minima have the same Numerical Value in a Symmetrical Parabolic Arch but with Change of Sign.*—For the maximum of tension or maximum in a member has the same numerical value, as for the maximum of compression or minimum. For fully loaded the extrados and oblique bars are zero in force diagram. Let the one side, as the left, be loaded, then they have some value. Now load the other or right side, they are again zero, that is, the action of the two loadings is equal and contrary. We have, therefore, in this case, only to obtain one of the maxima on any given bar and the other is given, being equal in value but opposite in sign. This is approximately true of a flat circular arch.



Section VIII.—Method of Determining the Stresses in the Parts of a Girder directly from the Shearing Force and Bending Moments.

84. *Example of Determining the Stresses in a Parallel Boomed Girder directly from Shearing Force and Bending Moment, figs. 66 and 67.*—We already have $Rr = M$, also $Qq = M$, or

$$\frac{M}{r} = R = \text{strain in boom } R$$

$$\frac{M}{q} = Q = \text{strain in boom } Q.$$

Now,

$$\text{Intercept on cord polygon} = \frac{M}{h}$$

so that if in constructing the cord polygon we choose as the value of h a multiple of

$$r \text{ or } q \text{ as } 2r, 3r, \dots nr$$

then intercept

$$= \frac{M}{nr} = \left(\text{in this instance } \frac{M}{3r} = \frac{R}{3} = \frac{\text{strain in boom}}{3} \right).$$

It is evident that the intercept must be measured on the force polygon scale.

The force polygon has, on its line of weights, three series of loads :—

First, or central series, structural weight of girder distributed over the nine joints.

Second, or upper series, weight of travelling load moving towards A ; taken from fig. 7.

Third, or lower series, weight of supplementary load moving towards B , the front buffers of the two engines touching each other.

In fig. 66, the foremost wheel of the travelling load is just over the sixth joint; $g(8)$, $g(3)$ are the verticals through the centres of gravity of the eight wheels of the travelling load and of the three wheels of the supplementary load respectively.

The cord polygon of the structural weight of girder, forms the arch-like figure. To it is joined, after the manner of fig. 16, the



travelling load and the closing line t_6 drawn; t_6 is transferred to the force polygon giving the two supporting forces $P_{A(6)}$ and $P_{B(6)}$.

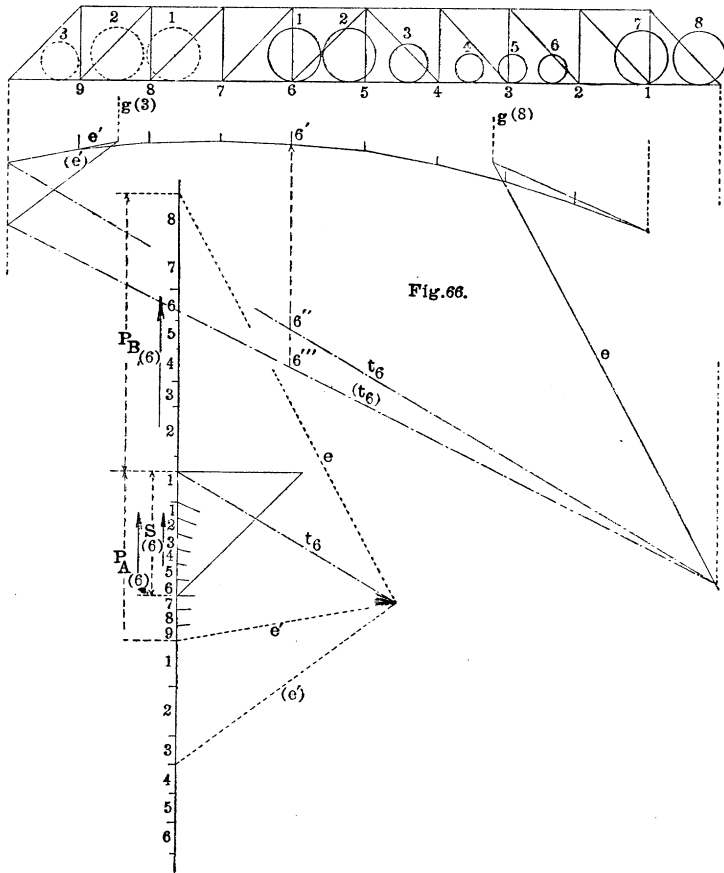


Fig. 66.

$$P_{A(6)} - (\text{structural weights } 7 + 8 + 9) = S_6 \\ = \text{shearing force in front of load.}$$

The intercept $6'6''$ = strain in booms $\div 3$ measured on weight scale. The cord polygon being formed by the ray e passing through the resultant of the load, and not by the separate weights of that load, the intercept (40) is not a correct measure of the bending moments to the right of the vertical at $6'6''$.

By means of the ray (e') we include the bending moment of the series of supplementary wheels, and obtain the maximum bending moment, *i.e.*, the maximum stress, $\div 3$, viz., $\overline{6''6''}$ for that joint (6).

In fig. 67. These operations have been repeated for every joint of the girder, and joining the points $6''5''4'' \dots$ and $6''5''4'' \dots$ we obtain by means of the latter line the maximum stresses on the booms. By means of the closing lines $t_9, t_8, t_7, t_6, t_5, t_4 \dots$ transferred to the force polygon we obtain the shearing forces at each joint, and thence $s_9, s_8, s_7, s_6 \dots$ the measures of the stresses in the diagonals.

We observe that from s_9 to s_3 inclusive, the shearing force being upwards \uparrow and arising from the reaction P_A of support A the stress in these bars following the notation of fig. 34 will be \diagup and in compression as far as s_6 , and thus suitable for transmitting the pressure to support A , and as the stress in bars 4 and 3 must be \diagdown they are unsuitable to transmit the pressure to A , wherefore the panels $\overline{5, 4}$ and $\overline{4, 3}$ must be counterbraced in order to transmit under compression the pressure to A .

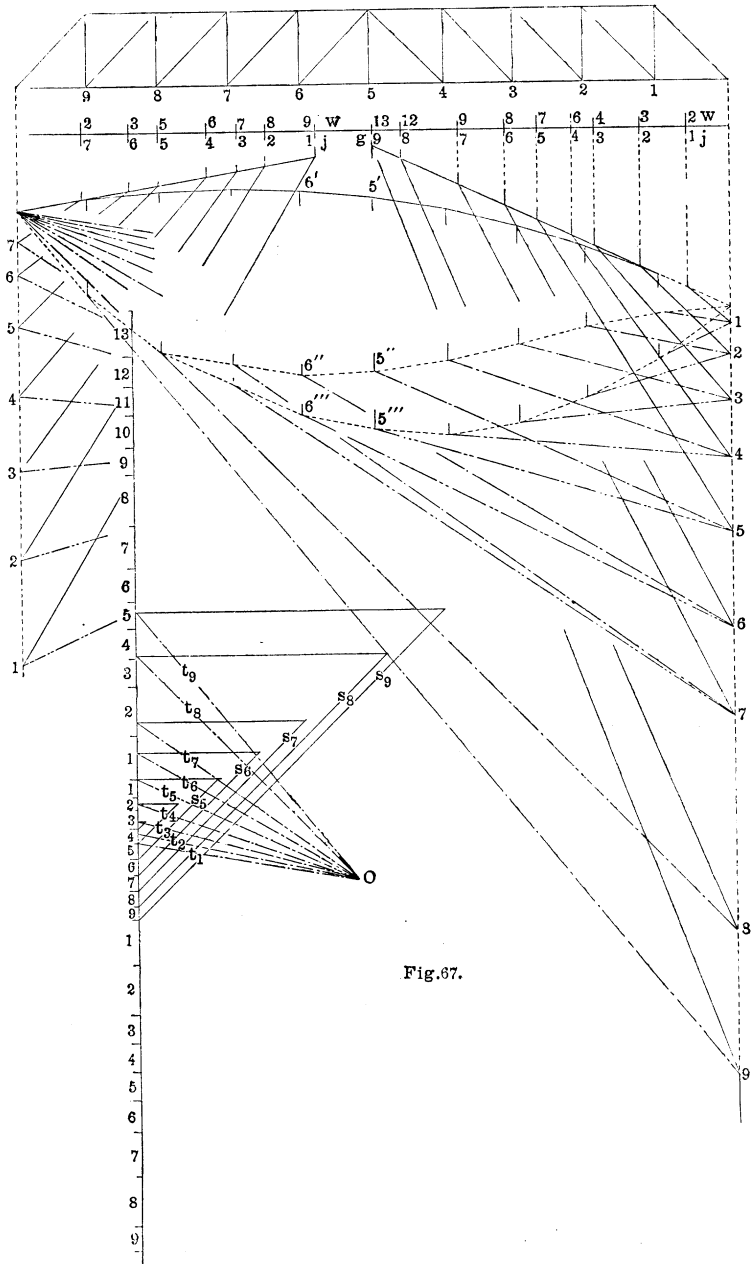
The strains appropriate for bars 1, 2, 3, 4, 5, 6 are those for bars 9, 8, 7, 6, 5, 4 already obtained.

The figures $\frac{\dots 7, 6, 4, 3, 2 w}{\dots 5, 4, 3, 2, 1 j}$ on the line g immediately below the girder are points in the verticals taken from line g , fig. 7, through the centres of gravity of 7, 6 \dots 3, 2 wheels of the travelling load, the front wheel of each group having arrived just over the 5th, 4th \dots 2nd, 1st joint of the girder. For the 8th and 9th joints however one wheel was rolled beyond the joint (art. 79, fig. 64).

Section IX.—Forms of Girders satisfying given Conditions.

85. *As the Stresses on the Diagonals of a Girder are dependent on the Curvature of the Booms, we may so construct them as to satisfy certain Conditions of Stress in the Diagonals.*

86. *The Parabolic Girder.*—Let the condition be given, that there be no stress upon the diagonals under a full load. Then the form to be given to the boom is that of the combined cord polygons of full load, for then the resultant of the pressures on



one side of any load go through the corresponding (QR) point of the beam and $S = 0$.

The girder shown on fig. 68 has been derived from the cord polygon of full load of fig. 67.

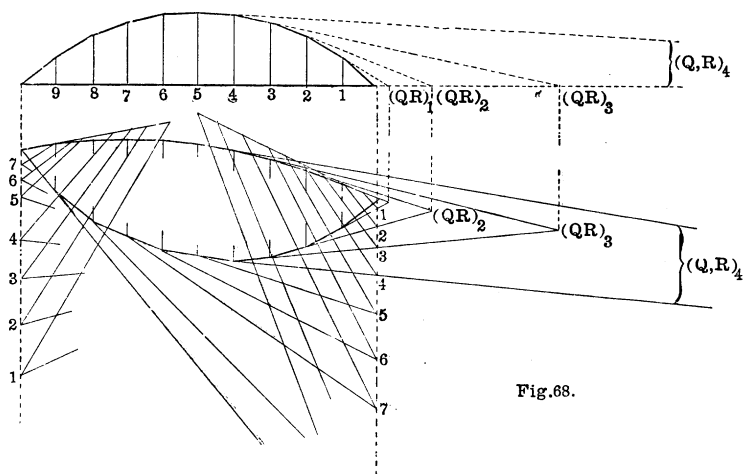


Fig. 68.

When the load is considered uniform or equally distributed over the joints, the form of the boom is parabolic (49).

87. *The Schwedler or Hyperbolic Girder.*—If we compare the symmetrical parallel-boomed girder with, say, the diagonals from the middle falling towards the points of support, the maximum in this case preponderates and is in compression, the minimum is in tension in all the panels, with the exception of the middle panels, which however may be put into tension by a sufficiently great increase of the permanent pressure. Hence we may expect an intermediate form to exist for which the *minimum of the diagonal stresses may be zero*.

In order to satisfy this condition, the resultant P of the pressures in front of the load must go through the point (QR) behind the load.

We have (fig. 69) taken the combined cord polygons of partial loading from fig. 67, and constructed the required girder, so that the resultants in that loading for part of the boom 1, 2, 3, go through the points $(Q, R)_1$ $(Q, R)_2$ $(Q, R)_3$. The same construction beyond joint 3 would give a cusp in the middle of the

boom, an impracticable construction, whence that part requires to be drawn in horizontally, and the condition cannot there be strictly fulfilled.

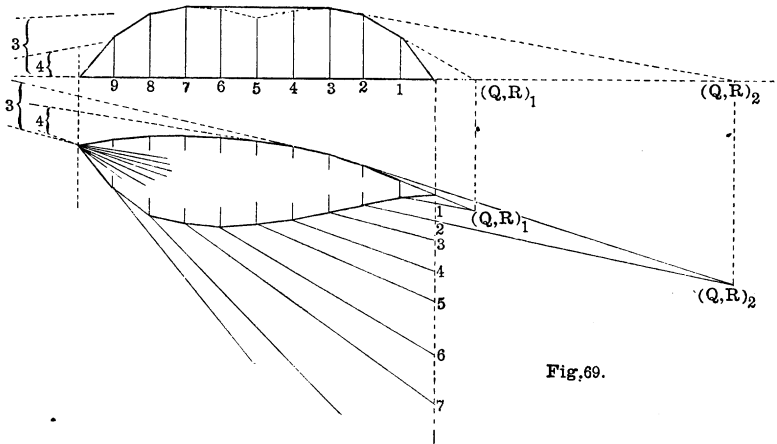


Fig. 69.

88. *Pauline Girder, or Bow and String Girder with equal Stress throughout the Boom.*—Having found the bending moment polygon for a full load, as in fig. 67, and decided on the central depth of the girder above ^{and} below the horizontal line *AB*, fig. 70. Let *M* be the central ordinate of the bending moment polygon, and *c* the central ordinate of the girder from the horizontal, then $\frac{c}{M}$ is a constant ratio.

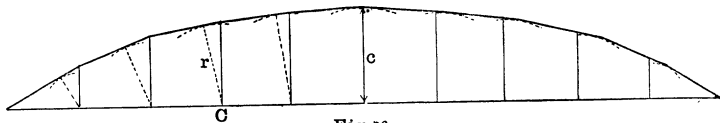
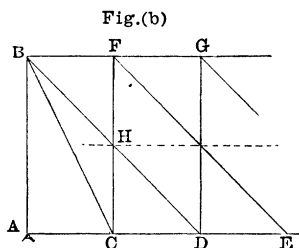
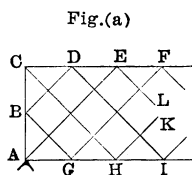


Fig. 70.

At any point *C* in the horizontal line of the girder, with $\frac{c}{M}$ × bending moment at *C* = *r* as radius, describe a small arc. Do the same at other points. Draw in the boom touching these arcs. It is unnecessary to give a formal demonstration.

For the principle which regulates the proportion between the depth *c* above and the depth *c'* below the horizontal line, see art. 103.

88(a). *Compound Girders*.—The stresses in compound girders may be resolved by apportioning the load over the joints of the boom and then resolving each system of triangles as a separate frame, the stresses in any bar common to several systems being added having regard to signs. Thus in the “*Lattice*,” fig. (a), we have four systems, $CHK \dots AEL \dots ABGF \dots ABDI \dots$. In the system composed of a *St. Andrew's Cross*, alternating with a vertical bar, there are two systems. In the system “*Linville*,” fig. (b), we have two systems, $ABCFE \dots$ and $ABDG \dots$.



88(b). *Position of Roadway*.—The sign of stress in any bar arising from travelling load varies according as the roadway lies upon the upper or lower boom, and the same bar in a compound system may have stress of one sign in one part and the contrary sign in the other when the roadway is laid along the central depth of the girder as shown by the dotted line in fig. (b), where FH is, under such circumstances, in tension, and HC in compression. In fact, the law connecting the lines of a node in a frame and the corresponding polygon in its force diagram, as elucidated in article 76, holds true universally, and weight of roadway enters as a line into that node through which the roadway passes.

*Section X.—Problems in Cranes combining several of the
Operations of the preceding Two Chapters.*

89. *Pivot Reaction and Force Diagram of a Fairbairn Crane*. Fig. 71 is the skeleton sketch of a Fairbairn crane. There are two points of support, A at the surface, B at the pivot. There

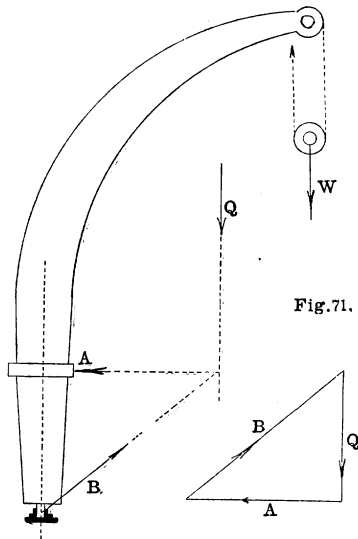


Fig. 71.

is a load W and its own weight w through G . The resultant of these two forces is Q . The two reactions must equilibrate Q and in order to this must meet in a point in its line of action. Now reaction at A is necessarily horizontal and meets the line of action of Q in point (A, Q) , wherefore reaction at pivot B must likewise meet Q in this point. We have now a node of three lines for our cord polygon, wherefore the triangle ABQ is our force polygon.

90. *Bending Moments and Stresses in Common Crane.*—Fig. 72 is the skeleton sketch of an ordinary crane. There are two points of support, one S at the surface of the ground, one P at the pivot. There are four holding down bolts, one pair at A and B , another pair at A' and B' . There is the force Q being the resultant of the load L and the structural weight W . Its line of action being obtained by means of force polygon, pole O , and corresponding cord polygon which can both be recognised in the figure. Q is distant q from the axis of the pillar, giving a moment $Q \cdot q$.

i. Transform the moment $Q \cdot q$ into $H \cdot h$ (43, 44) see (i.) in figure.

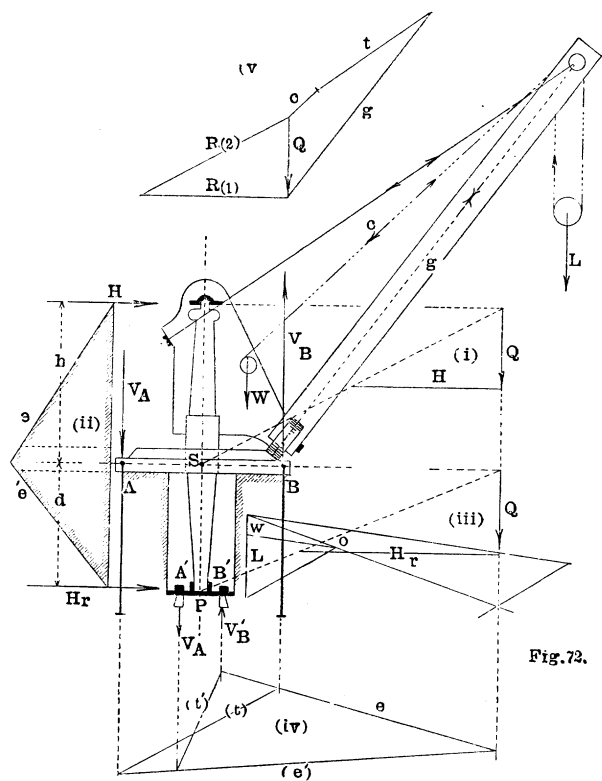
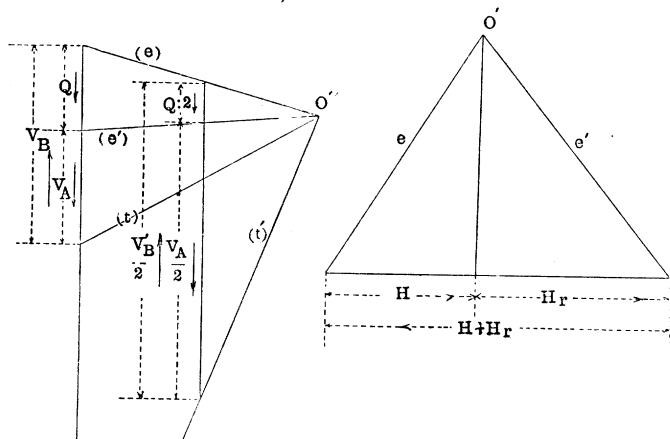


Fig. 72.



ii. $H \cdot h$ is the bending moment of the pillar at the surface S and H is the force producing bending moment on the pillar above S .

iii. $H_r \cdot d$ is the reaction moment (30) equal and opposite to $H \cdot h$. The value of H_r may be obtained either by the completion of the force polygon O' to the force H , and the three points, the point of action of H and the two reacting points S and P , or directly by transforming $Q \cdot q = H_r \cdot d$ (43, 44), see iii. in figure.

iv. H_r is the force producing bending moment from S to P and the cord polygon corresponding to the O' , force polygon marked (ii) in figure, and hatched measures the bending moment : pole distance, in pillar.

v. For the reactions V_A, V_B in the bolts at A and B , we have the downward force Q with its line of action, and the two points of support A and B , whence we can construct the force polygon O'' and its corresponding cord polygon, marked (iv.), whose rays are (e) , (e') , and (t) . The reaction V_A puts the bolt at A in tension.

vi. For the reactions V'_A, V'_B at A' and B' , we have again the downward force Q with its line of action, and the two points of support A' and B' . The pole O'' and the extreme rays (e) and (e') of force and cord polygons serve also here, but the closing ray (t') is not the same. On account of want of space $\frac{Q}{2}$ has been employed giving $\frac{V'_A}{2}$ and $\frac{V'_B}{2}$.

vii. The figure marked (v.) gives the stresses in t , c , and g , the reaction at S and the oblique reaction at the pivot (R_1 and R_2).

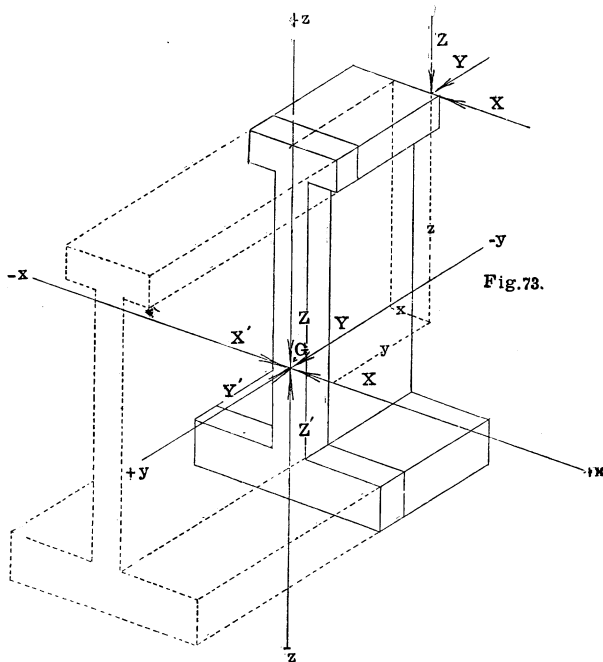
CHAPTER III.

THE BEAM.

Section I.—General View of the Forces Applied and Resisting in a Beam.

91. *Definition of a Beam.*—The uniform beam is generated by the motion of a given cross section, in a direction at right angles to its plane.

The centre of gravity of this cross section generating at the same time a line called the axis of the beam.



92. *The Exterior Forces applied to a Beam.*—By the parallelogram of forces (see any work on statics) any force P applied to a beam, fig. 73, can be resolved into three forces X , Y , Z at right angles to one another and acting upon the same point.

Any number of forces $P_1, P_2, P_3 \dots$ the coordinates of whose points of action from a convenient set of axes, are $x_1y_1z_1, x_2y_2z_2, x_3y_3z_3 \dots$ acting upon a rigid body, may by the extension of the method of (25) be replaced by three forces $\Sigma X, \Sigma Y, \Sigma Z$, acting at any point G and three couples

$$M_x = \Sigma(Zy - Yz) \text{ acting around axis of } x.$$

$$M_y = \Sigma(-Xz + Zx) \quad \quad \quad \text{ " } \quad \quad \quad y,$$

$$M_z = \Sigma(Yx - Xy) \quad \quad \quad \text{ " } \quad \quad \quad z,$$

moments tending to turn the couple from

x to y ,

y to z ,

z to x ,

being reckoned positive.

93. *Resisting Stresses in a Beam.*—Let fig. 73 represent a beam having the applied forces transferred from their points of action x, y, z to the centre of gravity G of the cross section, and let the point G be the origin of coordinates.

In considering this beam, we suppose the applied forces $P_1, P_2 \dots$ so small that they occasion no distortion of form in the beam.

The exterior or applied forces may be concentrated on certain points of a beam, but the resisting stresses are distributed throughout its whole material.

For the sake of greater simplicity of notation we will denote $\Sigma X, \Sigma Y, \Sigma Z$ by X, Y, Z , and consequently keep out the symbol Σ in the notation of the couples. To the longitudinal force Y at the centre of gravity G of the cross section, there is an equal and opposing stress which under the hypothesis that Y is too small to occasion distortion, is distributed equally over the cross section. Let A denote the area of the cross section, p_y the longitudinal stress per unit of area then

$$p_y = \frac{Y}{A}.$$

94. *Nomenclature of the Stresses in a Beam.*—The three forces X, Y, Z are resisted by the reactions of the beam, Z is the usual

vertical shearing force of chapter I. (25—28), X is an unusual transverse horizontal shearing force, Y is a longitudinal horizontal force, resisted by the reaction of some obstacle abutting against the end of the beam.

The manner in which the opposing shearing stresses are distributed over the longitudinal and cross sections of the beam will afterwards appear.

Of the three couples $M_x = Zy - Yz$ is the usual bending moment around the axis of x with which we are familiar. $M_y = -Xz + Zx$ is the moment of torsion, which ought never to occur in well designed structures, but becomes of importance in the dynamics of machinery. $M_z = Yx - Xy$ is an unusual bending moment around the vertical axis z .

95. *Hypothesis in regard to the Stresses in a Beam.*

i. The applied forces are so small, that they cause no alteration in the form of the beam.

ii. That the material of the beam is stressed in proportion to its distance (a) from a certain longitudinal line, in case of torsion, (b) from a certain longitudinal plane in the case of the other two couples M_x and M_y , having a line for its trace both in longitudinal and cross sections.

This line is called the neutral line or axis, and the plane the neutral plane.

Let m be an indefinitely small area of a cross section (fig. 74*b*, p. 117). Call it an area particle and at a distance z from the neutral axis then the stress is proportional (ii) to mz . If p be the stress per square unit induced by any one of the couples and at distance unity from the neutral axis, then the stress on any area particle m is pmz , and the sum of these stresses over the area of the cross section is

$$p \cdot \Sigma mz.$$

Section II.—Beam with Normal Forces only in the Plane of Symmetry.

96. *Conditions necessary to Equilibrium between Forces and Stresses when only Normal Forces act on a Symmetrical Beam, and in the Plane of Symmetry.*

i. Forces in this case—

$$Y = 0, X = 0$$

$$M_z = 0, M_y = 0$$

$$x = 0.$$

We have therefore only remaining the direct vertical force

$$Z$$

and the couple

$$M_x = Zy.$$

ii. The first condition necessary to equilibrium in this case (a) that the neutral axis pass through the centre of gravity of the cross section.

We know from statics, that for perfect equilibrium to exist among a number of forces at any point the sum of these forces must be zero, and their couples zero.

The forces here to be equilibrated are

$$Z \text{ and } Zy.$$

Of these Z generates shearing stress on the beam equal and opposite to itself (distributed in the interior of the beam in a manner afterwards to be elucidated) and the turning moment Zy generates the opposing stress $p \cdot \Sigma mz$, the stress at any particle being (90) pmz , and acting with an arm z , whence

$$Zy + p\Sigma mz^2 = 0,$$

and, as a couple cannot be equilibrated by a force, the forces in this couple must be in equilibrium among themselves, *i.e.*

$$p\Sigma mz = 0 \text{ wherefore } \Sigma mz = 0,$$

which means, that the neutral axis passes through the centre of gravity G of the cross section.

iii. The second condition necessary to equilibrium is (b) that the neutral axis must have no tendency to rotate around G into a new position, *i.e.* the forces excited in the beam must not form a couple around G . These forces are pmz and their leverages around G are x ; we must therefore in order to equilibrium, have

$$p\Sigma mzx = 0, \text{ or } \Sigma mzx = 0.$$

Now in the case we are here considering, the axis of symmetry is vertical and the resultant of forces is in this axis, so that a horizontal line through G alone fulfils the condition required for the neutral axis, for if any element m be taken whose co-ordinates are $+x, +z$, another element m can be found on the other side of this axis whose coordinates are $-x, +z$, whence

$$\Sigma mx = 0$$

and the condition necessary to equilibrium is fulfilled by the neutral axis in the cross section going through G , and at right angles to the axis of symmetry.

97. *Radius of Gyration.*— Σmz^2 is known as the *moment of inertia* of the cross section. If A be the area of the cross section $= \Sigma m$, and k such a quantity, that

$$k^2 A = \Sigma mz^2,$$

k is known as the radius of gyration.

98. *Fundamental Formula for Stress in Outer Filament of Beam in this Case.*—If the outer filament of the beam above and below the neutral axis be distant from it c and c' respectively (fig. 74*b*, page 117) and as

$$p = \frac{Zy}{\Sigma mz^2},$$

and

$$pc = \frac{c \cdot Zy}{\Sigma mz^2}, \quad pc' = \frac{c' \cdot Zy}{\Sigma mz^2},$$

or in words

$$\text{Stress in outer filament of beam} = \frac{c \times \text{bending moment}}{\text{moment of inertia}}$$

a fundamental formula.

99. *Comparison between Formula for Stress in this Case and Stress in Booms of Open Girder.*—In open girders the resistances to bending have been reduced to direct tensile and compressile stresses throughout the sectional area of the boom, by means of the formula

$$\text{Stress on unit of area} = \frac{\text{bending moment}}{\text{depth of girder} \times \text{area of boom}}.$$

This formula, though so unlike that in (98) is essentially the same for (fig. 76).

Calling $c + c' = v$ and A , the area of one of the booms, then the ordinary formula is

$$pc = \frac{Zy}{v \cdot A} \text{ and } pc' = \frac{Zy}{v \cdot A},$$

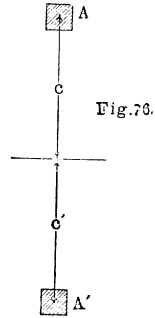
and applying the formula of (98), viz.

$$pc = \frac{cZy}{\Sigma mz^2}, \quad \Sigma mz^2 \text{ becomes } c^2A + c'^2A'.$$

Then when $c = c'$ and $A = A'$

$$pc = \frac{c \cdot Zy}{2 \cdot c^2A} = \frac{Zy}{2cA} = \frac{Zy}{vA}$$

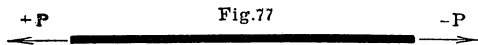
the same as the ordinary formula.



100. *Modulus of Elasticity*.—The formula $Zy = p \cdot \Sigma mz^2 = pI$, is only true on the principle of infinite rigidity; we go therefore to unfold a formula which takes elasticity into account—

$$Zy = \frac{EI}{r}.$$

Let a bar, (fig. 77), be drawn in the direction of its length by a force P equally distributed over the cross section of the bar. Let



L be its original length in feet,

l its elongation in feet under

P pounds strain,

K its sectional area in square inches,

$$p = \frac{P}{K} = \text{pounds strain per square inch.}$$

Experiment has shown that

l is directly proportional to the force P ,

„ „ to the length L ,

and inversely proportional to the section K ,

i.e.

$$l = C \cdot \frac{P \cdot L}{K}, \quad . \quad . \quad . \quad (1)$$

or

$$\frac{l}{L} = C \cdot \frac{P}{K} = C \cdot p, \quad (2)$$

where C is a constant.

Calling

$$C = \frac{1}{E} \quad . \quad . \quad . \quad (3)$$

gives

$$\frac{l}{L} = \frac{p}{E} \quad . \quad . \quad . \quad (4)$$

that is, if we conceive the material to retain its elasticity through such an extent that after receiving an elongation

$$l = L, \quad . \quad . \quad . \quad (5)$$

it would on withdrawal of P recover its original length, E would be the force per square inch of section necessary to effect it.

E is called the modulus of elasticity, and is found by experiment.

101. *Radius of Curvature of a Beam, and Differential Equation of Deformed Neutral Axis.*—From the value of E for any material, we are able to find the line formed by the neutral axis under bending forces.

Taking the equation (4)

$$\frac{l}{L} = \frac{p}{E}$$

and changing the symbols to those of fig. 74,

$$\frac{ds}{s} = \frac{p}{E}, \quad . \quad . \quad . \quad (6)$$

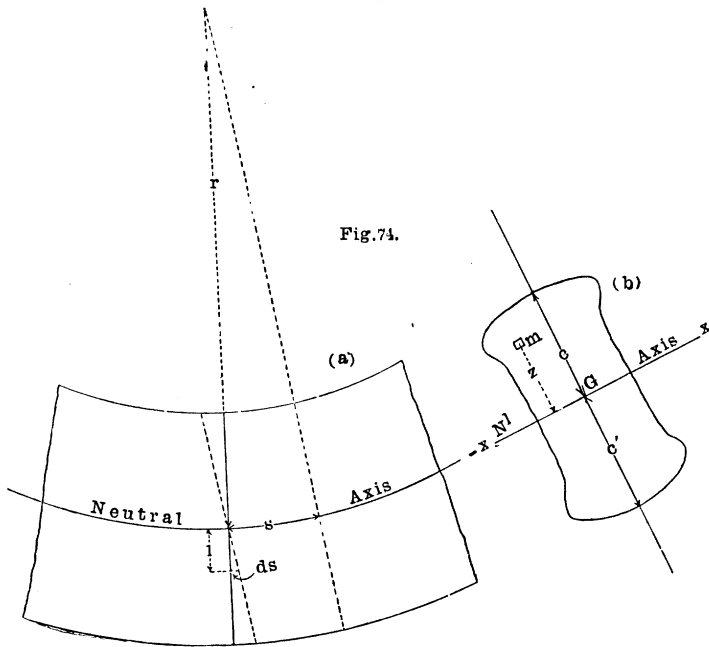


Fig. 74.

but the similar triangles which have the sides r and s in the one corresponding to 1 and ds in the other, give us

$$\frac{ds}{s} = \frac{1}{r} \quad \dots \dots (7)$$

From (6) we have

$$p = E \frac{ds}{s} \quad \dots \dots (8)$$

substituting, we have

$$p = \frac{E}{r} \quad \dots \dots (9)$$

From (93)

$$Zy = p \Sigma mz^2 \quad \dots \dots (10)$$

substituting

$$= \frac{E}{r} \cdot \Sigma mz^2 \quad \dots \dots (11)$$

calling Σmz^2 by the symbol I

$$Zy = \frac{EI}{r} \quad \dots \dots (12)$$

where Zy is the bending moment,
 E , the modulus of elasticity,
 I , the moment of inertia,
 r , the radius of curvature of the beam.

If we take the neutral axis before deformation as axis of abscissa, y being an abscissa, z an ordinate, we will have, by the well known differential formula

$$r = \frac{\left\{ 1 + \left(\frac{dz}{dy} \right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2z}{dy^2}} \quad . \quad . \quad (13)$$

The smallness of the deformation permits us to neglect

$$\left(\frac{dz}{dy} \right)^2$$

in comparison with unity, whence we have approximately

$$\frac{1}{r} = \frac{d^2z}{dy^2} \quad . \quad . \quad . \quad (14)$$

and equation (12) becomes

$$Zy = EI \cdot \frac{d^2z}{dy^2} \quad . \quad . \quad . \quad (15)$$

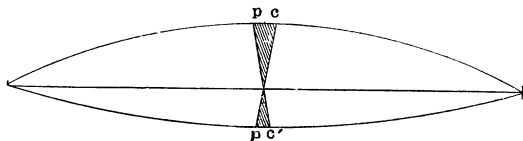
and is the differential equation of the deformed neutral axis of a beam lying upon its supports.

102. *Geometrical Signification of $\frac{d^2z}{dy^2}$.*—Let a, b, c be three ordinates to a curve (fig. 75), at a very small distance h from each other we have, for successive differences,

| | | |
|----------|---------|--------------|
| Ordinate | ' | " |
| a | | |
| | $b - a$ | |
| b | | $c - 2b + a$ |
| | $c - b$ | |
| c | | |

103. *Ratio of Distances of Upper Arched Boom and Lower Arched Boom from the Common Horizontal Chord in the Pauline Girder.*—We interpolate this article without formal demonstration.

Fig. 78



pc is the unit of stress (fig. 78) in the upper boom, pc' that in the lower boom. A and A' the areas of the upper and lower booms respectively. E and E' the moduli of elasticity of their materials respectively. Then, as the horizontal tension generated by the upper boom must be equilibrated by the horizontal compression generated by the lower boom, A and A' must be proportional directly to pc and pc' and inversely as E and E' . Let C be a constant introduced to produce equality, we have


$$A = C \cdot \frac{pc}{E} \text{ and } A' = C \cdot \frac{pc'}{E'}$$

or

$$c = \frac{AE}{AE + A'E'}(c + c'), \quad c' = \frac{A'E'}{AE + A'E'}(c + c').$$

104. *Graphical Construction of Moments and Products of Inertia, in the Case of a System of Heavy Particles.*

i. Let (fig. 79, p. 123) 1, 2, 3, 4, 5 be a number of heavy particles with weight v corresponding to 1, 2, 3, 4, 5 on the line of weights of the force polygons, having a pole distance b . By means of two force polygons as in (19), we find through the intersection of two axes X and Z the centre of gravity G of these particles. Extending the rays of the two cord polygons to these two axes, we find (4) $v_1x_1 : b, v_2x_2 : b \dots$ for the value of the intercepts on the Z axis, and $v_1z_1 : b, v_2z_2 : b \dots$ on the X axis.

ii. Call $v_1x_1 : b, v_2x_2 : b \dots$ a new line of weights for the same particles, choosing any pole distance c (pole O') from whence to form a new cord polygon. This new cord polygon will have an  form, and extending its rays to the axis Z we find for the value of the intercepts on this axe $v_1x_1^2 : bc, v_2x_2^2 : bc$ for by similar triangles, *e.g.*

$$c : \frac{v_1x_1}{b} :: x_1 : \frac{v_1x_1^2}{bc}$$

and the sum of these intercepts is

$$\Sigma_1^5 vx^2 : bc.$$

This expression is the *moment of inertia* : bc of the heavy particles around Z .

It is seen from the formation of this second cord polygon that the values of $vx^2 : bc$ are wholly additive, and this agrees with analysis, for x^2 is always positive. It is also worthy of note that the first and last sides of the cord polygon are parallel, for the first and last rays of the force polygon c coincide, for

$$v_1x_1 + v_2x_2 + v_3x_3 + v_4x_4 + v_5x_5 = 0,$$

Z going through the intercept of the extreme rays; a similar procedure would give us

$$\Sigma_1^5 vx^2 : bc,$$

but this is not given on the figure.

iii. Turning round the force polygon c (pole O') until its line of weights is parallel to the X axis (see pole (O')), we find for the value of the intercepts on this axe

$$v_1x_1z_1 : bc, v_2x_2z_2 : bc \dots$$

for by similar triangles, *e.g.*

$$c : \frac{v_1x_1}{b} :: z_1 : \frac{v_1x_1z_1}{bc}$$

and the sum of these intercepts is

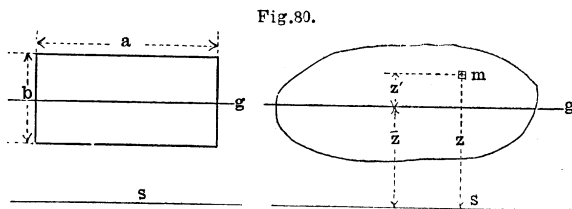
$$\Sigma_1^5 vxz : bc.$$

This expression is the *product of inertia* : bc of the heavy particles around the centre of gravity G .

iv. In this illustration, no constant a becomes introduced, for the heavy particles have been considered to be without dimensions. With regard to the constant b the rule (6) must be attended to, that is, should the intercept GZ or GX be measured on the scale of distance, b must be reckoned on the scale of weights, and *vice versâ*.

The same remarks apply to the constant c .

105. *Theorem. Moments of Inertia of a Lamina about any Axis.*—The moment of inertia of a lamina about any axis is equal to the moment of inertia of the lamina about that axis, conceived concentrated at its centre of gravity *plus* its moment of inertia around a parallel axis through its centre of gravity.



Let s (fig. 80) be the axis around which the moment of inertia of the lamina is required. Let g be a parallel axis through the centre of gravity of the body

z , the ordinate of an area particle m from axis s

z' , the ordinate of an area particle m from axis g

\bar{z} , the ordinate of the centre of gravity of the lamina from axis s .

Then by algebraic multiplication

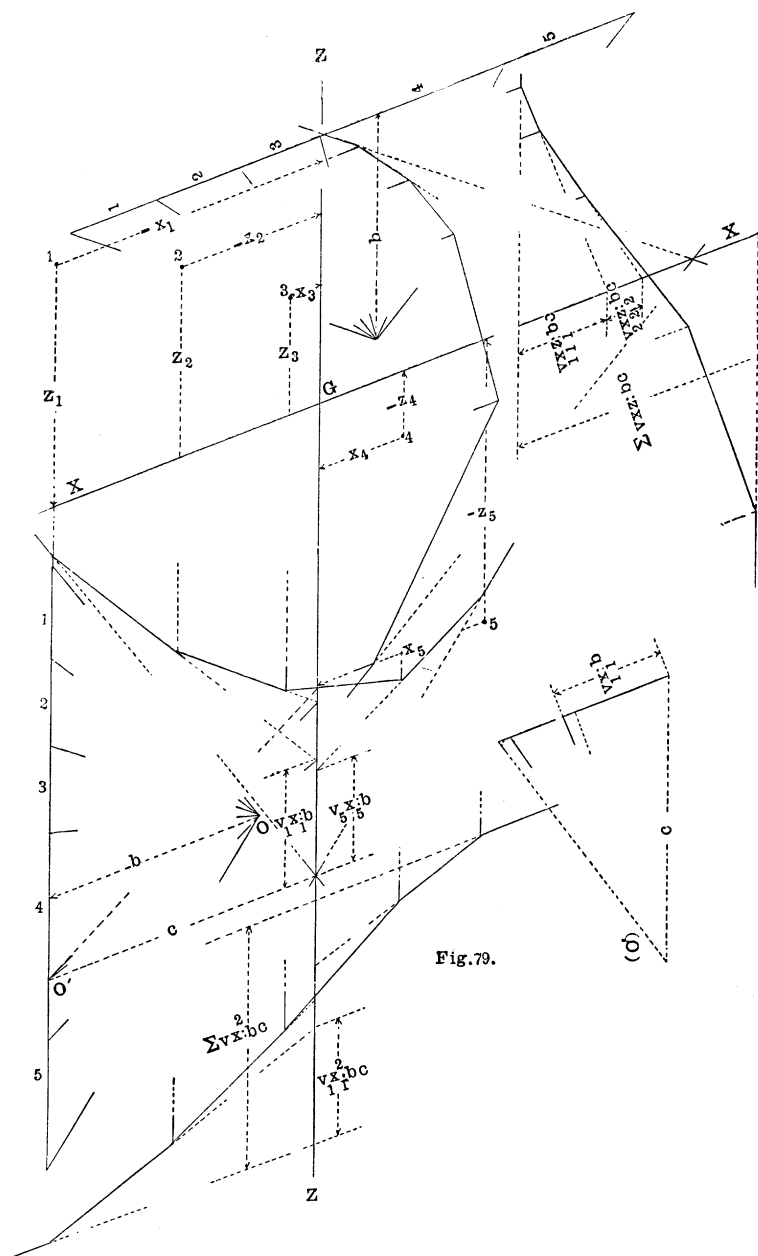
$$\begin{aligned}\Sigma m z^2 &= \Sigma m (\bar{z} + z')^2 \\ &= \Sigma m (\bar{z}^2 + 2\bar{z}z' + z'^2)\end{aligned}$$

but as

$$\Sigma m z' = 0, \quad 2\Sigma m \bar{z} z' = 0,$$

whence

$$\Sigma m z^2 = \Sigma m (\bar{z}^2 + z'^2). \quad \text{Q.E.D.} \quad (1)$$



106. *Theorem.* (i.) *The Moment of Inertia, and* (ii.) *Radius of Gyration of a Rectangular Lamina, around an Axis g going through its Centre of Gravity and Parallel to One of its Sides a, the other Side being b.*

i. The general expression Σmz^2 becomes

$$a \int_{-\frac{b}{2}}^{+\frac{b}{2}} z^2 \cdot dz = \frac{ab^3}{12} \quad . \quad . \quad . \quad (2)$$

ii. The equation for the radius of gyration k

$$k^2 A = \Sigma mz^2$$

becomes

$$k^2 ab = \frac{ab^3}{12} \text{ or}$$

$$k^2 = \frac{b^2}{12} = (0.289b)^2 =$$

$$\text{or} \quad k = 0.289b = \sqrt{\left(\frac{1}{12}\right)} \cdot b \quad . \quad . \quad . \quad (3)$$

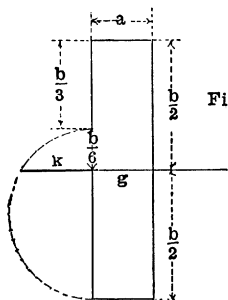


Fig. 80. a.

iii. This admits of easy graphical construction. As $\frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$; upon $\frac{2}{3}b$ describe a semicircle (fig. 80a) and upon the diameter of this semicircle at a distance $\frac{1}{6}b$ from its extremity erect a perpendicular. The semicircle and diameter cut off k . (*Eucl.* ii. 14.)

107. *Graphical Construction of the Moments of Inertia of a Rectangular Lamina, around any axis parallel to one of its Sides.*—Let it now be required to obtain graphically the value of the analytical expression (equa. 1, art. 105)

$$\Sigma mz^2 = \Sigma m(\bar{z}^2 + z'^2)$$

for a rectangle $ef = \Sigma m$.

We have seen that

$$\Sigma mz'^2 = k^2 \Sigma m = k^2 \cdot ef,$$

whence

$$\Sigma mz^2 = (\bar{z}^2 + k^2) \Sigma m, \quad . \quad . \quad . \quad . \quad (4)$$

polygon, from which, with a pole distance b , constructing the corresponding cord polygon we obtain from the intercept of the extreme rays on s the statical moment : ab , for, by similar triangles

$$b : \frac{ef}{a} :: \bar{z} : \frac{\bar{z} \cdot ef}{ab}.$$

ii. Again, constructing a second force and cord polygon with

$$\frac{\bar{z} \cdot ef}{ab}$$

for the line of weights and any given pole distance c and the vertical through the end point of

$$\bar{z} + \frac{k^2}{\bar{z}}$$

for the angular point of the cord polygon we have in the same manner

$$c : \frac{\bar{z} \cdot ef}{ab} :: \bar{z} + \frac{k^2}{\bar{z}} : \frac{(\bar{z}^2 + k^2)ef}{abc}.$$

This is the expression (equa. 4) for the moment of inertia : abc of the rectangular lamina ef , around an axis s parallel to the side f .

Note, m will be retained to represent an area particle, A a whole area, whence $\Sigma m = A$, and \bar{A} a definite measurable part of A whence $\Sigma \bar{A} = A$.

108. *Graphical Construction of the Combined Moments of Inertia and Combined Radius of Gyration of several Rectangular Laminae.*—Let it now be required to find the moment of inertia and radius of gyration of several rectangular laminae, so placed that the axis around which the moments are measured goes through the centres of gravity of them all, and is likewise parallel to a side in each of them.

Let g (fig. 82) be the axis around which the combined moments of inertia and combined radius of gyration of the rectangular laminae $\bar{A}_1, \bar{A}_2, \bar{A}_3$ are required.

Erect k_1, k_2, k_3 , the radii of gyration of $\bar{A}_1, \bar{A}_2, \bar{A}_3$ respectively

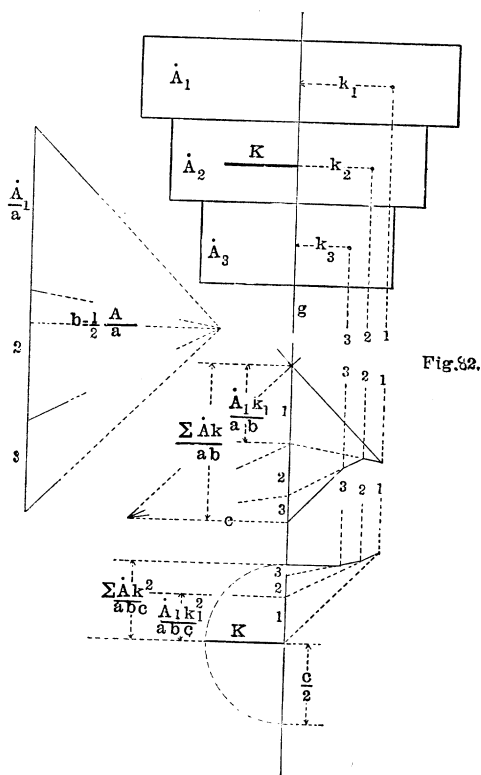


Fig. 32.

upon this axis. Reduce the laminæ to lines by some base a (15) and with

$$\frac{\dot{A}_1}{a}, \frac{\dot{A}_2}{a}, \frac{\dot{A}_3}{a}$$

for the line of weights in a force polygon, and with a pole distance b (not necessarily $= \frac{1}{2} \cdot \frac{A}{a}$) form a cord polygon with parallels to g through the distances k_1 , k_2 , k_3 , to determine its angular points, and extend its sides till they meet the axis g . The intercepts on this axis measure, by similar triangles

$$b : \frac{\dot{A}_1}{a} :: k_1 : \frac{\dot{A}_1 k_1}{ab}$$

and the sum of the intercepts is therefore

$$\frac{\Sigma Ak}{ab}.$$

If $b = \frac{1}{2} \cdot \frac{A}{a}$ then

$$\frac{\Sigma Ak}{ab} = \frac{\Sigma Ak}{\frac{1}{2}A}.$$

Consider these intercepts as a new line of weights, and taking any pole distance c (let it be for convenience laid off from an extreme end of the intercepts) form a second cord polygon having the distances k_1, k_2, k_3 , for the distances of its angle points from the axis g , and extend the sides till they meet this axis.

The intercepts on this axis measure, by similar triangles

$$c : \frac{A_1 k_1}{ab} :: k_1 : \frac{A_1 k_1^2}{abc}$$

and the sum of the intercepts is consequently

$$\frac{\Sigma Ak^2}{abc}.$$

If $b = \frac{1}{2} \cdot \frac{A}{a}$ then

$$\frac{\Sigma Ak^2}{abc} = \frac{\Sigma Ak^2}{\frac{c}{2}A}.$$

Add $\frac{c}{2}$ to the length of the intercept sum and find a mean proportional K to

$$\frac{c}{2} \text{ and } \frac{\Sigma Ak^2}{\frac{c}{2}A}$$

(*Euclid*, ii. 14) i.e., find K such that

$$K^2 = \frac{\Sigma Ak^2}{\frac{c}{2}A} \cdot \frac{c}{2} = \frac{\Sigma Ak^2}{A},$$

whence

$$K^2 A = \Sigma Ak^2,$$

whence K thus found is equal to the radius of gyration of the sum of the rectangular laminæ around the axis going through their centres of gravity and parallel to a side in each of them.

109. *Graphical Construction of the Moments of Inertia and Radii of Gyration of the Cross Section of a Beam.*—Let us apply the foregoing graphical constructions to finding the moments of inertia and radii of gyration of the cross section of the beam (fig. 83, p. 131).

As the beam has an axis of symmetry, one operation according to (14) suffices to obtain the centre of gravity G of the cross section.

We have A_1, A_2, A_3 the areas of the upper flange, the web, and the lower flange of the beam respectively, and

$$\frac{A_1}{a}, \frac{A_2}{a}, \frac{A_3}{a}$$

their respective lengths (marked 1, 2, 3, on the figure) on the line of weights of the first force polygon $z_1, z_2, -z_3$, the respective distances of their centres of gravity from G .

Extend the rays of the cord polygon to meet the line through G . Construct the distances from G , viz. :—

$$z_1 + \frac{k_1^2}{z_1}, \quad z_2 + \frac{k_2^2}{z_2}, \quad z_3 + \frac{k_3^2}{z_3}$$

(In our figure the points z_1 and z_3 practically coincide with

$$z_1 + \frac{k_1^2}{z_1} \text{ and } z_3 + \frac{k_3^2}{z_3},$$

so we will employ the indicated construction (107) to the web z_2 only.)

Then with the intercepts on the axis through G and any pole distance c , construct the second \smile formed cord polygon, the sum of whose intercepts is (107, ii.)

$$\frac{\Sigma m z^2}{abc} = \frac{\Sigma A(z^2 + k^2)}{abc} = \frac{\Sigma A(z^2 + k^2)}{a \frac{1}{2} \frac{A}{a} c} = \frac{\Sigma A(z^2 + k^2)}{\frac{c}{2} A}, \quad \dots \quad (1)$$

then finding (108) a mean proportional

$$K = \sqrt{\frac{c}{2} \cdot \frac{\Sigma A(z^2 + k^2)}{\frac{c}{2} A}} = \sqrt{\Sigma(z^2 + k^2)} \quad \dots \quad (2).$$

Again, for the value of K' around the axis of symmetry erect upon this axis k'_1, k'_2, k'_3 obtaining K' exactly as in (108), k'_2 is not marked on figure but its extremity is marked by 2'. The Σm , i.e. the ΣA force polygon, has to be turned round 90° in order to carry out the construction (19).

Note.—If the moment of inertia alone is required, it is well to take for pole distances two numbers whose value is 10 or $10 \times 10 \dots$ or even 10 or 2×10 or 3×10 , for then the numerator in the fraction (equa. 1) being the moment of inertia, measuring the corresponding intercept sum on the appropriate scale, we can easily multiply this intercept by the numbers corresponding to the pole distances.

110. *Apportioning the Area of a Beam to the Bending Moment.* We are now in a position to deal with the strength of a beam to sustain a given bending moment Zy .

Recurring to the equation

$$\begin{aligned} Zy &= -p \cdot \Sigma z^2 m \\ &= -p K^2 \cdot \Sigma m = -p \cdot K^2 A, \end{aligned}$$

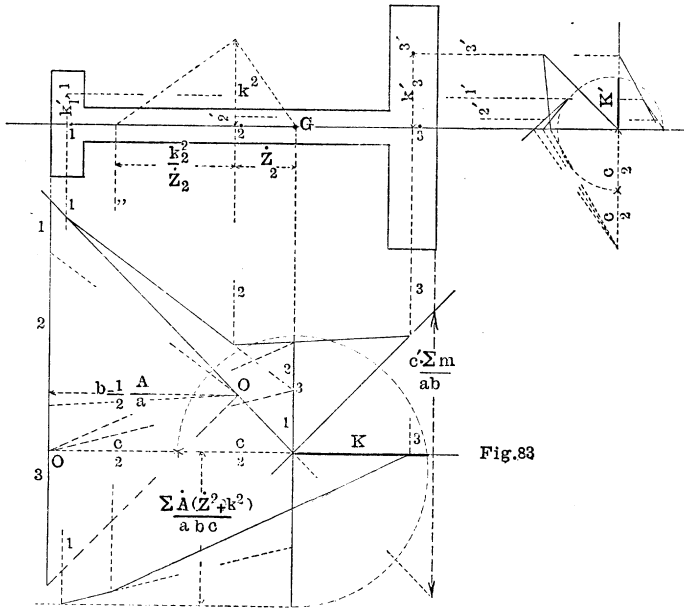
and to equation

$$pc = \frac{c \cdot Zy}{\Sigma m z^2} = \frac{c Zy}{K^2 A},$$

pc and Zy being given, we can apportion the area A of a beam by approximate trials until the second member gives the required value to pc .

111. *Moments of Inertia and Radii of Gyration in a Rail.*—Plate I. represents, upon the cross section of a rail, the same operations as have been performed upon the beam, fig. 83, and described in (109). The cross section of the rail being divided into many more ΔA portions, there being only three ΔA or A portions in the beam, while there are thirteen ΔA in the rail. It consequently gives to the student a fuller view of the previously described constructions.

The ΔA in the rail being so narrow, \dot{z} and $\dot{z} + \frac{k^2}{\dot{z}}$ in each of them have been taken as practically coinciding in obtaining the moment of inertia and the value of K around the X axis. For the value of the moment of inertia and of K' around the Z



axis, the operations on the beam and on the rail are those of (108).

112. *Distribution of Shearing Stress on the Longitudinal Section of a Beam.*—Let fig. 84 be a graphical representation of the tensions and compressions in a beam at the points y and $y - \Delta y$ at all ordinates z measured from the neutral axis of the beam, viz. pz and $p'z$, by means of two pairs of triangles. Let fig. 84a represent the two pairs of triangles superimposed, forming a new pair of triangles from the difference of breadth in the first two pair, the breadth of this new pair at any ordinate z will measure $\Delta p \cdot z$.

Now suppose the beam compressed above and extended below the neutral axis, then at any layer it would be less compressed in the new section $\left| \rightarrow \leftarrow \right|$ giving as the figure is drawn an unbalanced force towards the left, to be resisted by cohesion between the layers, the resistance being shearing stress. On the other side of n at the new section there is an unbalanced force toward the right $\leftarrow \left| \rightarrow \right|$.

Let us now interpret our equations with the help more especially of plate I. and of figs. 83 and 84.

Let $Zy = p \cdot \Sigma mz^2$ (plate I.) and let us consider a , b , and c to have been so chosen that their continuous product $abc = 1$, then the intercept between the extreme rays of the second cord polygon on the X axis measures Σmx^2 .

The extreme rays e and e' of the second cord polygon are the lines of action of the two equal and opposite forces $+p$ and $-p$ of the couple of which Σmx^2 is the lever arm, resisting Zy . Again, at the new section, e and e' are the lines of action of $+p'$ and $-p'$ with the same lever arm Σmz^2 resisting $Z(y - \Delta y)$.

Between these two sections there is a residual couple, but we will deal best with these couples by turning them round through a right angle (44).

Take any side $\overline{3, 4}$ of the second cord polygon (plate I.), this side is, of course, the resultant of all the forces on either side of it. Extend it till it meets the two extreme rays e and e' and project it orthogonally on one of them. Call this projection h_3 .

Consider now, the two similar and corresponding triangles formed by the second force and second cord polygons respectively, viz.

side c of the one, and side h_3 of the other,

hypoth. $O_c(3, 4)$ of the one and hyp. $\overline{3, 4}$ (extended) of the other, we have the following proportion

$$\frac{\Sigma_1^3 x \cdot z \cdot \Delta z}{ab} : c :: \frac{\Sigma mz^2}{abc} : h_3,$$

that is

$$h_3 = \frac{\Sigma mz^2}{\Sigma_1^3 x \cdot z \cdot \Delta x} = \frac{K^2 A}{\Sigma_1^3 x \cdot z \cdot \Delta z} \cdot \cdot \cdot \cdot \cdot \cdot \quad (1)$$

Again, where $\overline{3, 4}$, extended, meets e and e' , are two points in the line of action of two resultants R_1^3 , R_4^{13} , which are necessarily equal and opposite. We have now turned the couple round a right angle, and its forces are now R_1^3 and R_4^{13} , while its lever arm is h_3 .

We have then, for the first section

$$Zy = R_1^3 \cdot h_3 = p \Sigma_1^3 x \cdot z \cdot \Delta z \times \frac{K^2 A}{\Sigma_1^3 x \cdot z \cdot \Delta z} = p K^2 A \cdot \cdot \cdot \cdot \cdot \quad (2)$$

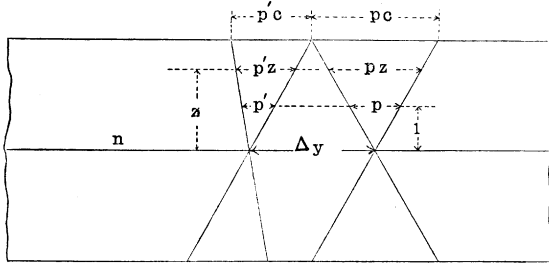
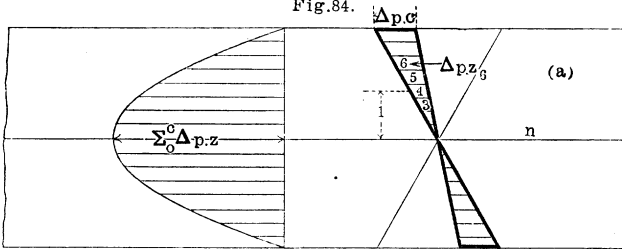


Fig.84.



For second section

$$Z(y - \Delta y) = R_1^3 \cdot h_3 = p' \Sigma_1^3 x \cdot z \cdot \Delta z \times \frac{K^2 A}{\Sigma_1^3 x \cdot z \cdot \Delta z} = p' K^2 A \quad (2a)$$

For the residual force between the two sections

$$Z \Delta y = \Delta R \cdot h_3 = \Delta p \Sigma_1^3 x \cdot z \cdot \Delta z \times \frac{K^2 A}{\Sigma_1^3 x \cdot z \cdot \Delta z} = \Delta p K^2 A \quad (2b)$$

If we had taken the side say $\overline{5, 6}$ we would have for the equation of the residual force

$$Z \Delta y = \Delta R_1^5 \cdot h_5 = \Delta p \Sigma_1^5 x \cdot z \cdot \Delta z \times \frac{K^2 A}{\Sigma_1^5 x \cdot z \cdot \Delta z} = \Delta p K^2 A,$$

whence the resultant force varies inversely as its lever arm, and is consequently the greater the nearer it is to the neutral axis, for the nearer the axis the steeper the side of the cord polygon.

The second cord polygon whose sides are $e, 3, 4, e'$, has its force polygon O_c for pole $O_c(e), O_c(e')$ for its extreme rays which coincide, and $O_c(3, 4)$ for its other rays. Its two impressed forces are $+R_1^3, -R_1^3$, and the horizontal tension introduced is

$\Delta p = c$, whence $O_c(3, 4)$ is the t ray of earlier constructions, and the vertical component of the stress is the intercept on the force polygon $\Delta p \Sigma_1^3 x . z . \Delta z$, which is therefore the vertical force at the joint 3 of the cord polygon, which can only be resisted by cohesion between the layers.

The following may assist the conception. Conceive the beam divided into imaginary layers coinciding with the extremities of the cord polygon sides and the cord polygon drawn upon it. Then each layer is a link in the cord polygon. Now take the links or layers $\overline{3, 4}$ and $\overline{2, 3}$, the tension on $\overline{3, 4}$ is the resultant of all the other tensions. The horizontal force is Δp and the vertical force is necessarily $\Delta p \Sigma_1^3 x . z . \Delta z$ and the resistance of the joint of the links to this force, that is the resistance to sliding between the two layers $\overline{2, 3}$ and $\overline{3, 4}$, is this vertical resistance = and opposite to

$$\Delta p \Sigma_1^3 x . z . \Delta z = \Delta R_1^3.$$

Returning to the equations 1 and 2b

$$\Delta R_1^3 = \Delta p \Sigma_1^3 x . z . \Delta z = Z \Delta y \quad . \quad . \quad (3)$$

This force $\Delta R_1^3 h_3$ is distributed over Δy of the longitudinal section; divide by Δy and we have the intensity per unit in length of longitudinal section

$$\frac{\Delta R_1^3}{\Delta y} . h_3 = Z, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and were it distributed, therefore, with that intensity along the longitudinal section for a length equal h_3 , it would equilibrate Z .

113. *Graphical Construction and Representation of the Distribution of Shearing Stress in a Beam.*—In the case of a beam of uniform breadth, the expression for the shearing stress over Δy of the longitudinal section, being

$$\Delta p \Sigma_z^c x . z . \Delta z$$

becomes

$$\Delta p . x . \Sigma_z^c . \Delta z,$$

for x being a constant may be taken out of the sign Σ of summation, and the law of distribution of shearing stress in this case can be represented by simple summation of $\Delta p \cdot z$ as in fig. 84, the curve formed by the ordinates assuming the form of the parabola.

Taking the equation

$$\Delta p = \frac{Z \Delta y}{K^2 A}$$

and substituting that value of Δp in the expression for shearing stress at any layer, we have

$$\Delta p \cdot \Sigma_z^c x \cdot z \cdot \Delta z = \frac{Z \Delta y}{K^2 A} \cdot \Sigma_z^c x \cdot z \cdot \Delta z = Z \Delta y \cdot \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}$$

The area over which this shearing stress is distributed is $x \cdot \Delta y$. Let σ be the shearing stress per unit of area, then dividing the above expression by $x \cdot \Delta y$ we obtain

$$\sigma = \frac{Z}{x} \cdot \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}$$

Writing down the following proportions—

$$\frac{\Sigma m z^2}{abc} : b :: Z : \frac{Z \cdot b \cdot abc}{\Sigma m z^2} = f \quad \dots \quad (1)$$

$$\begin{aligned} c : \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{ab} :: \frac{Z \cdot b \cdot abc}{\Sigma m z^2} : \frac{Z \cdot b \cdot abc \cdot \Sigma_z^c x \cdot z \cdot \Delta z}{abc \cdot \Sigma m z^2} \\ : \frac{Z \cdot b \cdot \Sigma_z^c x \cdot z \cdot \Delta z}{\Sigma m z^2} \\ : \frac{Z \cdot b \cdot \Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A} = S \cdot b \quad (2) \end{aligned}$$

When $b = \frac{1}{2} \frac{A}{a}$ this expression becomes

$$Z \cdot \frac{A}{2a} \cdot \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}$$

It will be observed that the last term of the first proportion is the third term of the second.

The fourth term of the second proportion is the expression for shearing stress per unit of length in the longitudinal

them off as ordinates from the axis of symmetry over their corresponding laminae, we obtain a number of points which, united by a line, we will call the curve of shearing stress.

But a is a base to which all our ΔA are reduced, and we shall find it convenient, instead of finding σ = shearing stress per unit of area, to find $\sigma \cdot ab$. Now

$$\sigma = \frac{S}{x}$$

and we have the proportion

$$x : a :: S \cdot b : \frac{S \cdot ab}{x} = \sigma \cdot ab,$$

or what is the same

$$\frac{x}{2} : \frac{a}{2} :: S \cdot b : \frac{S \cdot ab}{x} = \sigma \cdot ab.$$

Lay off (fig. 85(a), horizontally x or $\frac{x}{2}$ and $S \cdot b$, and vertically a or $\frac{a}{2}$. Complete the figure as shown, by forming two similar right-angled triangles, and we have graphically solved the above proportion. This construction has been carried out (fig. 85(a) separately for the value of $S_b b$.

It will be found also on plate I. at ΔA_6 , where

$$\frac{x}{2}, \frac{a}{2}, \sigma_6 ab, \text{ i.e. } \sigma_6 a \frac{1}{2} \frac{A}{a} = \sigma_6 \frac{A}{2} \text{ and } S_6 b, \text{ i.e. } S_6 \frac{A}{2a}$$

are marked.

114. *Recapitulation of Formulae*.—Distance of outer filaments of beam from neutral axis

$$c \text{ and } c'.$$

Stress on outside filament per unit of area

$$pc \text{ and } pc'.$$

Stress on filament at distance z from axis per unit of area

$$pz.$$

Stress on whole breadth and depth of filament at z

$$pz \cdot x \Delta z.$$

Moment of whole stress at z

$$pz \cdot x \Delta z \cdot z = px \cdot z^2 \cdot \Delta z.$$

Summation of moments of stress around neutral axis

$$p \Sigma_{-c}^{+c} x \cdot z^2 \Delta z.$$

Proceeding to another cross section distant Δy from the last.

Bending moment at first cross section

$$Zy = -p \cdot \Sigma_{-c}^{+c} x \cdot z^2 \Delta z.$$

Bending moment at second cross section

$$Z(y - \Delta y) = -p' \cdot \Sigma_{-c}^{+c} x \cdot z^2 \Delta z.$$

Force to be equilibrated by the beam between the sections

$$Z \cdot \Delta y = -\Delta p \cdot \Sigma_{-c}^{+c} x \cdot z^2 \cdot \Delta z.$$

Longitudinal shearing stress at any ordinate z between the two sections

$$S_z \Delta y = Z \Delta y \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit in length of longitudinal section at any ordinate z

$$s_z = Z \cdot \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}.$$

Intensity of shearing stress per unit of area in longitudinal section at any ordinate z_1

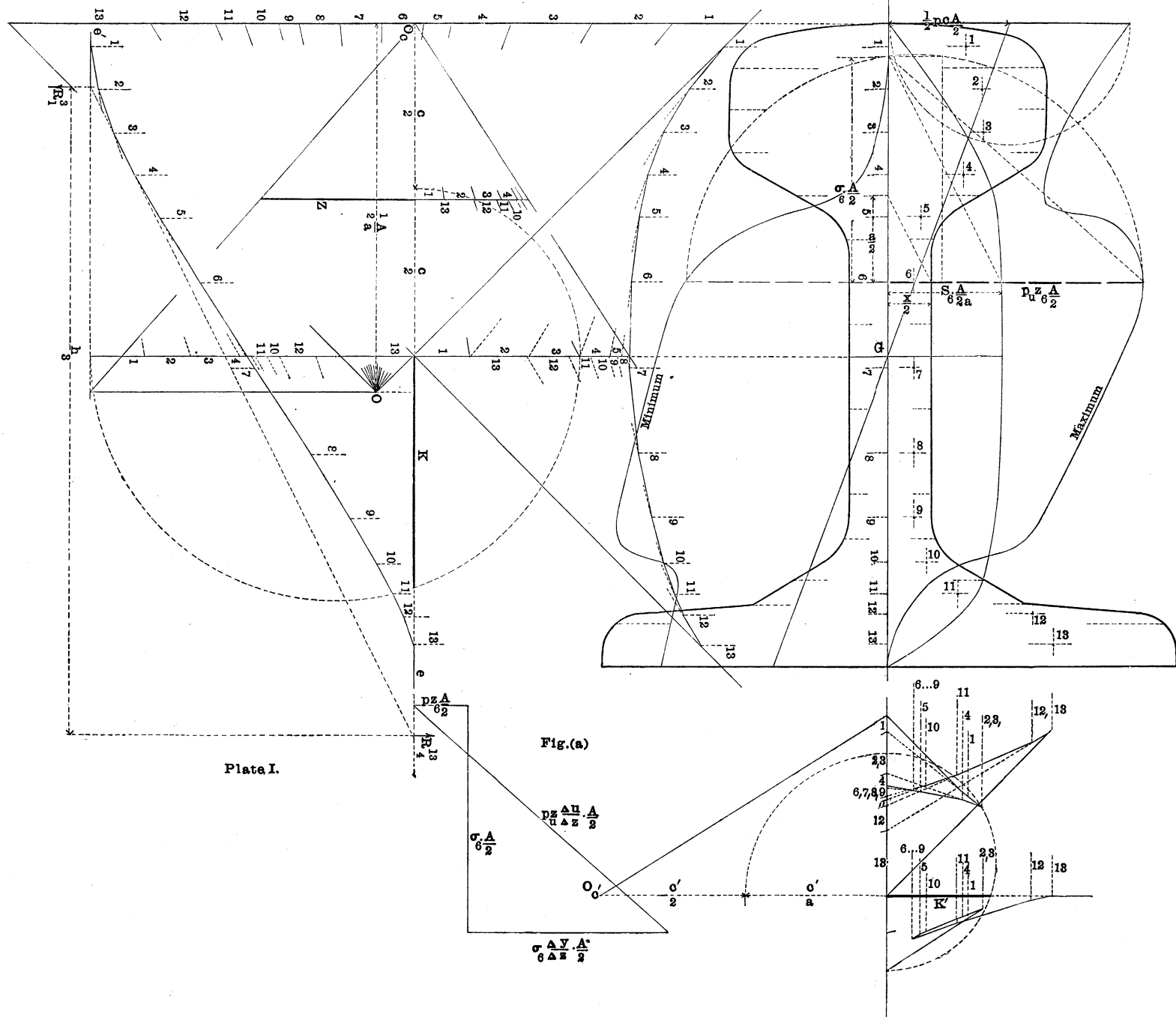
$$\sigma_z = \frac{Z}{x_1} \cdot \frac{\Sigma_z^c x \cdot z \cdot \Delta z}{K^2 A}.$$

Remember that in the foregoing

$$x \cdot \Delta z = \Delta A = A.$$

Section III.—Beam with Oblique Forces P in the Plane of the Vertical Axis of Symmetry.

115. *Effect of Oblique Force in the Plane of Symmetry in removing the Neutral Axis parallel to itself from the Centre of Gravity of the Cross Section.*



and where this expression equals zero is the line of neutral axis, viz. where

$$p\bar{z} = -p_y$$

whence

$$\bar{z} = -\frac{p_y}{p}.$$

116. *Point of Action in Cross Section of Resultant of Stresses.*—Let z unaccented be the value of z measured from the new neutral axis (fig. 86), then

$$z = \bar{z} + z',$$

and the stress on any element is now

$$mpz = mp(\bar{z} + z'),$$

and the sum of the moments is (105)

$$\begin{aligned} p\Sigma mz^2 &= p\Sigma m(\bar{z}^2 + z'^2) = p(k^2 + \bar{z}^2)\Sigma m \\ \frac{\Sigma \text{ Moments of Stresses around New Neutral Axis}}{\Sigma \text{ Stresses}} &= \frac{p\Sigma mz^2}{p\Sigma mz} \\ &= \frac{p\Sigma mz^2}{\bar{z}\Sigma m} = \frac{k^2}{\bar{z}} + \bar{z}. \end{aligned}$$

The distance from the neutral axis at which $p\Sigma mz$ acts is therefore

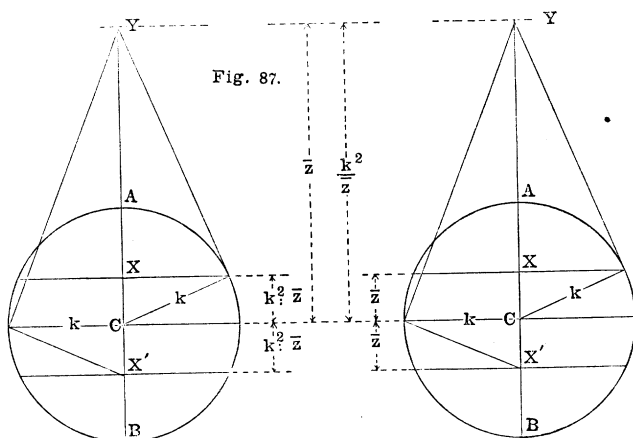
$$\frac{k^2}{\bar{z}} + \bar{z}.$$

117. *Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are Pole and Antipolar.*—Noting the construction of fig. 87. Y is the pole of the line n through X and making $CX' = CX$. X' is the antipole of the line n , the line n' going through X' is the antipolar of Y , and (*Proj. Geom.*)

$$-\frac{AX}{AY} = \frac{BX}{BY} \text{ or } \frac{AX}{BX} = \frac{AY}{BY}$$

or substituting their values in terms of k and \bar{z} marked upon them, then by similar triangles

$$\bar{z} : k :: k : \frac{k^2}{\bar{z}}$$



or

$$CY = \frac{k^2}{z},$$

and

$$\frac{k - z}{\frac{k^2}{z} - k} = \frac{k + z}{\frac{k^2}{z} + k}$$

and

$$X'Y = z + \frac{k^2}{z},$$

wherefore if n' going through X' is the neutral axis, the point of action of the forces goes through Y .

118. *Point of Action of Resultant of Stresses and Trace of Neutral Axis in Cross Section are interchangeable.*—Inversely as pole and polar are interchangeable (*Proj. Geom.*) if Y is in the neutral axis, X' becomes the point of action of the forces. For the same equation still holds good, for, exchanging the symbols, we obtain as formerly

$$X'Y = z + \frac{k^2}{z}.$$

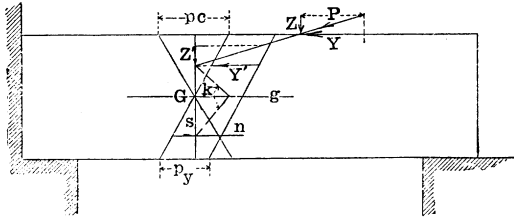
119. *Point of Action of Forces when Neutral Axis is removed to Edge of Cross Section.*—If by means of force Y of such a value

(a) If the neutral axis is at the outer edge of the cross section, its antipole to a circle with radius K around G is a point in the line of action of the resultant forces.

(b) If the point in the line of action of the resultant forces (in the plane of symmetry being understood) is at the outer edge of the cross section, the neutral axis is the antipolar of that point to a circle with radius K around G .

120. *Finding by means of Theorem of 117 the Neutral Axis at a given Section due to a given Oblique Force P .*—Fig. 89 represents longitudinal section of a beam, acted upon by an oblique force P in the plane of symmetry. Required its influence on the position of the neutral axis of the beam at some section s .

Fig. 89.



Decomposing P into Z and Y we have, Z the usual vertical shearing force at s , Y the longitudinal compressive force resisted by an obstacle at the end of the beam, P . $r = Zy - Yc$ the bending moment around G ; r being the lever arm of P around G .

Produce P till it intersect s . Through G draw the radius of gyration, k of the beam at right angles to s , and from point (P, s) draw a line to the extremity of k , and to that line from end point of k draw another at right angles cutting s in a point of n . (117.)

The action of the force P has now been decomposed into the shearing force $Z' = Z$, and the bending moment Yz , z being now measured from n to point (P, s) , the point of application of $Y' = Y$.

121. *The Neutral Axis in an Arch, or Retaining Wall.*—The preceding theorems (115—120) find their most important application in the theory of the arch and retaining wall.

When we come to treat of the arch we shall find that for any cross section of it (as s fig. 89), we are able to find (tentatively, no doubt), the oblique pressure P at that section and its point of action (P, s) in the section, from whence after (120) we can find the neutral axis for that cross section.

Now as an arch ought to be wholly in compression, the neutral axis n must lie without the cross section (for the stresses are on the one side of n in compression and on the other in tension), whence, after (119) the point of action of P must be within the two points X' and X'' (fig. 88), for if it be outside of these points the neutral axis is within the cross section.

Keeping in view the construction of fig. 80a and fig. 88, and remembering that the angle within a semicircle is a right angle, we can see that in an arch with a rectangular cross section, as in a stone arch, the pressure P must fall within the central third of the depth of the arch stone, in order that the whole depth of the arch stone may be in compression. This agrees with the old-established maxim. (122, 2.)

122-124. Geometrical Construction of DURAND-CLAYE'S Hyperbolas.¹

122. *Problem. To Construct the Values Y of Stress in a Rectangular Cross Section, it may be of an Arch, for every point $\frac{k^2}{z}$ of its application under Constant Stress of Outside Laminae.*—Let the line s (fig. 90) be the elevation of a rectangular cross section under constant stress u of outside lamina on the most compressed side, required Y for every point $\frac{k^2}{z}$ of application. Let v be the distance from u to point of application of Y .

1. When $v = c$ ($= \frac{1}{2}$ depth of rectangle) Y goes through G , the neutral axis is at infinity, Y is distributed equally over the cross section, and $Y = 2cu$.

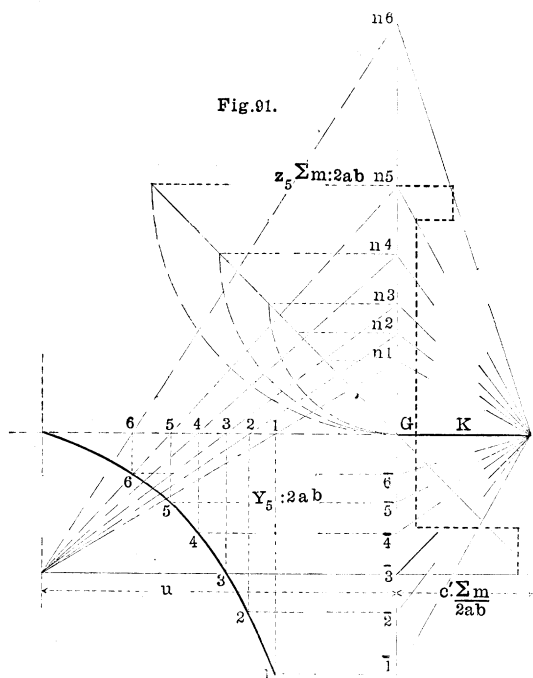
2. When $v = \frac{2}{3}c$ ($= \frac{1}{3} 2c$) Y goes through the point X' (119, fig. 88) at the edge of the core, and the neutral axis is on the outside edge of the cross section. The one outside lamina is

¹ Analytically treated by M. Durand-Claye in *Ann. d. ponts et Chauss.* 1867 and 1868.

as the representative lines of their areas to the base $2c$, we may transfer them to the lines of action of the forces they represent. This has been done for a number of points $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}$ from whence the hyperbolic curve $5, 4, 3, 0, 1, 2$ has been constructed, showing the values of Y to the base $2c$.

123. *Extension of Preceding Problem to Symmetrical Beam.*—

The construction of (122) can be extended to any symmetrical beam, *e.g.* that of fig. 83.



Let u be the stress per unit of area which shall act on the outside layer of the beam at a distance c' from the neutral axis, then (fig. 86) for any neutral axis distant z from G

$$p = \frac{u}{c' + z};$$

and as

$$p\Sigma mz = p \cdot \bar{z} \cdot \Sigma m,$$

z being measured from this neutral axis, we have

$$\frac{u}{c' + \bar{z}} \cdot \Sigma mz = Y = \frac{u}{c' + \bar{z}} \cdot \bar{z} \cdot \Sigma m = Y,$$

$$u \cdot \Sigma mz = Y(c' + \bar{z}) \quad \text{or} \quad u \cdot \bar{z} \cdot \Sigma m = Y(c' + \bar{z})$$

and as the product of the extremes is equal to the product of the means, we may write this last expression

$$(c' + \bar{z}) : u :: \Sigma mz : Y \quad \text{or} \quad (c' + \bar{z}) : u :: \bar{z} \cdot \Sigma m : Y.$$

Now (fig. 83), at a distance c' from G the intercept of the extreme rays of the first cord polygon

$$= c'\Sigma m : ab$$

whence the following construction.

Take (fig. 91) and at a distance c' from G lay off

$$c'\Sigma m : 2ab$$

at right angles to axis of symmetry and join its free extremity with G , and produce it, we have thus a line whose ordinate for any abscissa \bar{z} is

$$\bar{z}\Sigma m : 2ab, \text{ and } b \text{ being } = \frac{1}{2} \frac{\Sigma m}{a};$$

this line shall make $\angle 45^\circ$ with that axis, whence the construction given, being based upon the relation

$$\bar{z} = \bar{z}\Sigma m : 2ab,$$

for any value of \bar{z}

$$(c' + \bar{z}) : u :: \frac{\bar{z}\Sigma m}{2ab} : \left(\frac{u}{c' + \bar{z}} \cdot \bar{z}\Sigma m : 2ab \right) \\ : (Y : 2ab).$$

The special choice for the constant b , viz. $\frac{1}{2} \frac{A}{a}$, has given to this construction a certain simplicity, for it gives

$$\bar{z} \cdot \frac{\Sigma m}{2ab} = \bar{z},$$

so that all values of $\bar{z}\Sigma m : 2ab$ fall on the line through G ,

We might take any other value for b , but the construction would want simplicity. The construction given,

$$Y : 2ab = Y : 2a\frac{1}{2}\frac{A}{a} = Y : A$$

is reduced to the construction of the preceding problem, article (122).

124. *The Locus of $Y : 2ab$ is an Equilateral Hyperbola* of which the vertical through G is the one asymptote, and a horizontal drawn through a point in the vertical at a distance $\frac{k^2}{c'}$, above G is the other asymptote.

For from the proportion

$$\bar{z} + c' : u :: \bar{z} : \frac{u\bar{z}}{\bar{z} + c'}$$

we have the expression for the values of $G\bar{1}$, $G\bar{2} \dots$ upon the horizontal through G equal to

$$\frac{u\bar{z}}{\bar{z} + c'}$$

and the expression for the rectangle formed by the asymptotes and perpendiculars upon them from any point of the curve is

$$\frac{u\bar{z}}{\bar{z} + c'} \cdot \left(\frac{k^2}{\bar{z}} + \frac{k^2}{c'} \right) = \frac{u\bar{z}}{\bar{z} + c'} \cdot \frac{k^2(\bar{z} + c')}{\bar{z}c'} = u \cdot \frac{k^2}{c'}$$

a constant.

This theorem might also be proved from the properties of pencils of rays in involution.

Section IV.—Case of an Unsymmetrical Beam, or of a Symmetrical Beam unsymmetrically placed, the Plane in which the Forces act going through the Centre of Gravity of the Beam.

125. *Condition of Equilibrium in case in which the Couples $Zy - Yz$ and $-Xz + Zx$ exist, but $Zx = 0$, i.e. in which the Resultant of X , Y and Z is in a Plane passing through the Centres of Gravity of the Cross Sections of the Beam.*—The vertical component of these three forces, X , Y , Z , which we shall for brevity

call Z , determines the direction in which the leverage z of the reacting stresses, pmz , must be measured, but the neutral axis from which z must be measured must have no tendency to rotate round G , that is

$$\Sigma mzx = 0.$$

It is evident that the axis of x fulfilling that condition, cannot be laid down without previous investigation, except in the case of Section II., viz. that of the beam symmetrical to its vertical axial plane, vertically loaded and vertically placed. In fig. 94, we see at once that a horizontal x axis would not fulfil the condition

$$\Sigma mzx = 0.$$

We must therefore extend our knowledge of the theorems upon the moments and products of inertia.

126. *Transferring the Axes around which the Moments and Products of Inertia are taken, parallel to themselves.*—Let x and z , fig 92, be the co-ordinates of a lamina, having any point O for origin, m any area particle,

$$\Sigma m = A = \text{area of lamina,}$$

$$\Sigma x^2 = a^2,$$

$$\Sigma z^2 = b^2,$$

$$\Sigma xz = C.$$

Let there be drawn a second pair of co-ordinate axes, parallel to the first, having any other point O' for origin, and let the co-ordinates of their origin be z_1 y_1 , then

$$x = x_1 + x',$$

$$z = z_1 + z',$$

$$\Sigma xzm = \Sigma (x_1 + x') (z_1 + z'),$$

$$= \Sigma (x_1 z_1 + z_1 x' + x_1 z' + x' z') m,$$

and remembering that x_1 , z_1 , are constants and can therefore be put outside of the sign of summation

$$\Sigma xzm = Ax_1 z_1 + z_1 \Sigma x' m + x_1 \Sigma z' m + \Sigma x' z' m.$$

Let the co-ordinate axes x' , z' go through the centre of gravity of the lamina, then

$$\Sigma x'm = 0 \text{ and } \Sigma z'm = 0,$$

and the equation becomes

$$\Sigma xzm = Ax_1z_1 + \Sigma x'zm,$$

$$\Sigma xzm = A(x_1z_1 + C'), \quad . \quad . \quad . \quad (3)$$

$$\text{or } C'A = \Sigma xzm - Ax_1z_1.$$

This result may be thus expressed. The difference between the products of inertia referred to any co-ordinate axes through any point O and when referred to parallel co-ordinate axes through the centre of gravity of the lamina is equal to the product of the co-ordinates of the centre of gravity with the area of the lamina.

Let us write

$$\Sigma x^2m = a^2A, \quad \Sigma xzm = C \cdot A, \quad \Sigma z^2m = b^2A,$$

$$\Sigma x'^2m = k^2A, \quad \Sigma x'zm = C' \cdot A, \quad \Sigma z'^2m = h^2A,$$

then

$$C \cdot A = C' \cdot A + Ax_1z_1,$$

$$\Sigma x'^2m = \Sigma (x - x_1)^2m$$

$$= \Sigma x^2m - 2x_1^2A + x_1^2A$$

remembering that $\Sigma x'm = 0$.¹

$$\Sigma x'^2m = \Sigma x^2m - x_1^2A,$$

we finally obtain

$$\Sigma x^2m = \Sigma x'^2m + x_1^2A$$

$$a^2A = k^2A + x_1^2A,$$

¹ $\Sigma (x - x_1)^2m = \Sigma (x^2 - 2x_1x + x_1^2)m$
 $= \Sigma x^2m - 2x_1\Sigma xm + x_1^2\Sigma m$
 $= \Sigma x^2m - 2x_1\Sigma (x_1 + x')m + x_1^2\Sigma m$
 $= \Sigma x^2m - 2x_1^2\Sigma m - 2x_1\Sigma x'm + x_1^2\Sigma m$
 $= \Sigma x^2m - 2x_1^2A - 0 + x_1^2A,$

the expression in the text.

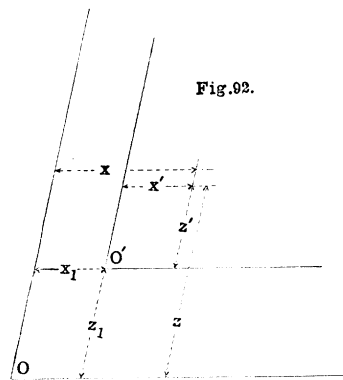


Fig. 92.

that is

$$a^2 = k^2 + x_1^2 \quad . \quad . \quad . \quad (1)$$

In the same manner

$$b^2 = h^2 + z_1^2 \quad . \quad . \quad . \quad (2)$$

Collecting the foregoing expressions we have

$$a^2 = k^2 + x_1^2 \quad . \quad . \quad . \quad (1)$$

$$b^2 = h^2 + z_1^2 \quad . \quad . \quad . \quad (2)$$

$$C = C' + x_1 z_1 \quad . \quad . \quad . \quad (3)$$

127. *Turning the Axes, in reference to which the Moments and Products of Inertia are taken, around their origin, gives an Ellipse for the locus of C, K and K' when their Values are laid down on these Axes.*

i. Knowing the moments and products of inertia

$$\Sigma x^2 m, \quad \Sigma z^2 m, \quad \Sigma x z m,$$

$$\text{or } a^2 A, \quad b^2 A, \quad C. A,$$

relatively to given axes GX and GZ (fig. 93) required

$$\Sigma q^2 m = h^2 A.$$

$\Sigma q^2 m$ being taken around an axis GQ going through the same origin Gq being measured as the ordinate from GQ in any given direction.

Substituting this value in $h^2 \Sigma m$

$$\begin{aligned} h^2 \Sigma m &= \Sigma q^2 m = \gamma^2 \Sigma x^2 m - 2\gamma \Sigma xzm + \Sigma z^2 m \\ &= \gamma^2 a^2 \Sigma m - 2\gamma C \cdot \Sigma m + b^2 \Sigma m, \end{aligned}$$

$$\text{or } h^2 = \gamma^2 a^2 - 2\gamma C + b^2,$$

$$\text{or } h^2 = \left(a\gamma - \frac{C}{a} \right)^2 + b^2 - \frac{C^2}{a^2}.$$

This last form being one by means of which h is easily constructed geometrically.

Lay off on GX and GZ , the radii of gyration $+a, +b, = GX$, and $GZ, -a, -b, = G(X)$ and GZ' and through end points of $+a, +b, -a, -b$, draw lines parallel to the axes $AA', AZ, A'Z', (A')(X)$, then GX, GQ cut off upon AA' the length

$$XQ = \gamma a.$$

Lay off from X

$$XO = \frac{C}{a}$$

and from O upon OX erect a perpendicular OH .

With the point X as centre and distance b , describe a semicircle AHA' . This will (*Euclid*, i. 47) cut off upon the perpendicular the length

$$OH = \sqrt{b^2 - \frac{C^2}{a^2}},$$

The distance $HQ = h$ (*Euclid*, i. 47).

By means of a semicircle DHD' from centre Q and radius h cut AA' in D and D' , and draw lines parallel to GQ .

For an indefinite number of positions given to Q upon the line AOA' as centres we obtain by means of semicircles $BHB' \dots$ an indefinite number of lines in pairs, parallel to each new position of GQ surrounding a form which they envelope.

When Q coincides with O ,

$$\gamma a = \frac{C}{a},$$

and

$$h^2 = b^2 - \frac{C^2}{a^2}.$$

ii. The punctuelle marked upon AA' is a punctuelle in involution of which O is the centre and OH the semi-axis (*Proj. Geom.*), and AA' , BB' , DD' . . . are pairs of corresponding points, and the forms which the pairs of parallels surround, is a conic section, and, being finite in all directions, is an ellipse. O being the centre of involution is the point which corresponds to itself, OA is therefore tangential to the ellipse at O , and GO is therefore the conjugate of GZ .

iii. *The Construction of the Principal Axes.*—Let GV , $G(U)V'$ be the principal axes, then VGV' is necessarily a right angle, whence, in order to obtain the two corresponding points V and V' , the centre Q of the necessary circle must be that point on the line AA' where $QG = QH$. To find that point is an elementary problem. It is the point where the perpendicular to GH from its middle point cuts OA .

iv. The co-ordinates of the point of contact O referred to the first axes are

$$a \text{ and } \frac{C}{a}.$$

Similarly the co-ordinates of the point of contact O' are

$$b \text{ and } \frac{C}{b},$$

and their product

$$a \cdot \frac{C}{a} = b \cdot \frac{C}{b} = C = \Sigma xz$$

is the same.

As we have made no hypothesis over the position and direction of the axes, the following result prevails generally.

That Σmxz in relation to two chosen axes is equal to the product of the co-ordinates of the end points of a conjugate diameter of x or z in the ellipse of inertia. If x and z are conjugate, then is $\Sigma mxz = 0$: for one ordinate,

$$\frac{C}{a} \text{ or } \frac{C}{b},$$

of the end-points equals zero.

v. The area of the triangle

$$GOX = \frac{a}{2} \cdot \frac{C}{a} = \frac{C}{2}.$$

The area of the triangle

$$GO'Z' = \frac{b}{2} \cdot \frac{C}{b} = \frac{C}{2},$$

whence it follows that $O'Z'$ is decreased in the same ratio that GZ' is increased or *vice versé*.

This is obtained by drawing OO' parallel to XZ' .

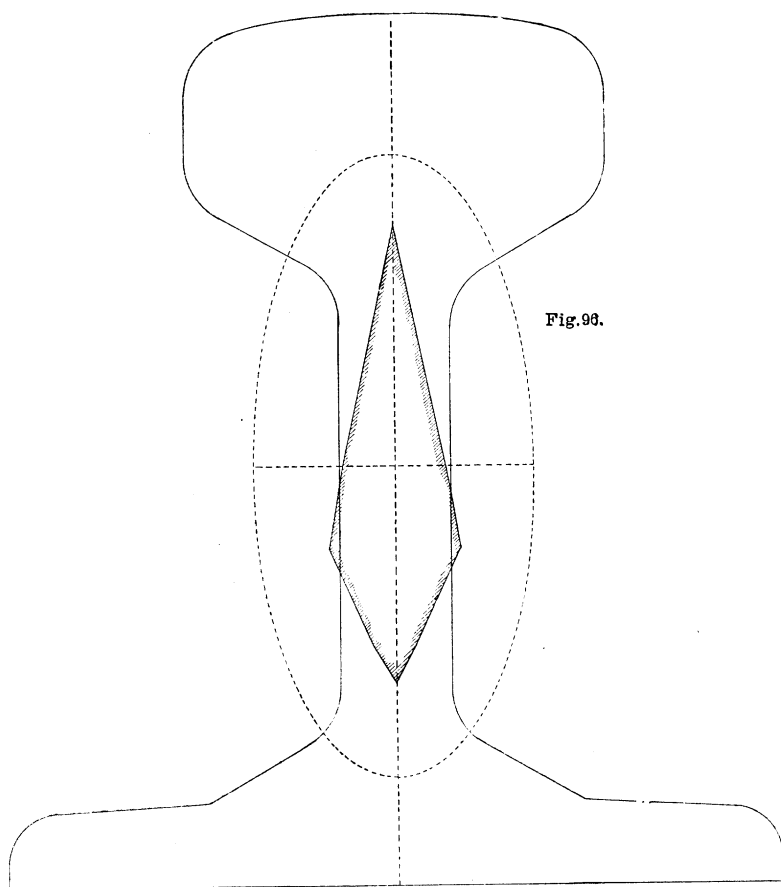
128. *When the Vertical Component Z (116) of the Applied Forces only exists, the Neutral Axis is the Diameter of the Ellipse of Inertia conjugate to that Diameter which, in the Cross Section, is the Trace of the Plane going through Z and the Centres of Gravity of the Cross Sections of the Beam.*—For, for that axis $\Sigma maz = 0$ (127, iv.).

We can now deal with the symmetrical beam unsymmetrically placed (fig. 94). Having found its moments and radii of gyration K and K' for the two principal axes for which $C = 0$, we can construct our ellipse of inertia, then the semi-diameter on ZG measures $h = \sqrt{\Sigma q^2}$ taken round its conjugate axis which is the neutral axis.

129. *If the Line of Action of the Resultant of Applied Forces passes through the Point Z, in the Trace of the Vertical Plane through ZG in which the Applied Force lies, then the Neutral Axis is removed to the Antipole of Z and lies parallel to the Conjugate Diameter of ZG, and conversely a line n' through Z parallel to the Conjugate Diameter is the Neutral Axis when the Point of Intersection of Antipolar with ZG is the Point of Action of the Forces.*—The method by which the antipolar to Z is obtained is obvious from the figure.

130. *Core or Heart of a Lamina.*—Suppose the neutral axis to pivot around the circumference of the cross section, then the point in the line of action of the forces which is always the antipole of the neutral axis to the ellipse of inertia traces the circumference of a body within the ellipse, called the *core* or

Fig. 96 is the cross section of the rail of Plate I., giving the core. It has been kept out of the plate to avoid confusion.



Section V.—Problems in connection with Moments and Products of Inertia.

131. *Having the Central Ellipse O of any Lamina, to construct the Moments and Products of Inertia of that Lamina around any other given Axes GX , GZ , and point G , figs. 97, 98, and 99.¹—*Equations (5 art. 107 and 3 art. 126) give us

$$\Sigma x^2 m = (\bar{x}^2 + k^2) \Sigma m = \bar{x}(\bar{x} + \frac{k^2}{\bar{x}}) \Sigma m \quad . \quad . \quad (1)$$

$$\Sigma xzm = (\bar{x}\bar{z} + C') \Sigma m = \bar{x}(\bar{z} + \frac{C'}{\bar{x}}) \Sigma m = \bar{z}(\bar{x} + \frac{C'}{\bar{z}}) \Sigma m \quad . \quad (2)$$

$$\Sigma z^2 m = (\bar{z}^2 + k^2) \Sigma m = \bar{z}(\bar{z} + \frac{k^2}{\bar{z}}) \Sigma m \quad . \quad . \quad (3)$$

where the ordinates of O , the centre of the central ellipse, are \bar{x} and \bar{z} .

Now, recalling the constructions in figs. 81 and 83, and comparing them with the form to the right of the above expressions, we see that we have $\bar{x}\Sigma m$ and $\bar{z}\Sigma m$ or the first moments. These of course are found in the line of weights of the second force polygons, and (dropping the accent of C for simplicity)

$$\bar{x} + \frac{k^2}{\bar{z}}, \quad \bar{z} + \frac{k^2}{\bar{z}}, \quad \bar{z} + \frac{C}{\bar{x}}$$

are the lever arms for the second moments and product of inertia, and must consequently meet us in the cord polygon of second moments.

Our problem consists then, first, in constructing the lengths of these lever arms.

i. To find

$$\bar{x} + \frac{k^2}{\bar{z}}$$

Draw tangents to the given ellipse parallel to GZ (their tangential points are the extremities of the diameter $= 2k'$). These tangents cut off the value $k = \gamma k'$ on a parallel to GX , and give us by construction of art. 129 (fig. 94) the distance from O on that parallel of $\frac{k^2}{\bar{x}}$.

¹ x_g, z_g of these and other figures is the \bar{x}, \bar{z} of the text.

Let $B'O = x'$, and by the same construction (129, fig. 94) obtain $\frac{k'^2}{x'}$, then the triangle whose sides are BO and $B'O$ is similar to the triangle whose sides are k and k' , whence

$$k' : k :: \frac{k'^2}{x'} : \frac{k^2}{\bar{x}},$$

whence we perceive that

$$\bar{x} + \frac{k^2}{\bar{x}}$$

is the lever arm for second moments of the lamina O around GZ .

It will be observed that the diameter parallel to GZ is conjugate to k' .

In the same manner we obtain (fig. 98)

$$\bar{z} + \frac{k^2}{\bar{z}}.$$

This has been given in a second figure to avoid confusion.

ii. To find

$$\bar{z} + \frac{C}{\bar{x}}, \text{ that is } \bar{z} + \frac{\Sigma xz}{\bar{x}}.$$

BG , we know (127, i., ii.), cuts off from the tangents parallel to GZ a length

$$= \frac{C}{\bar{k}},$$

and we have the following proportion

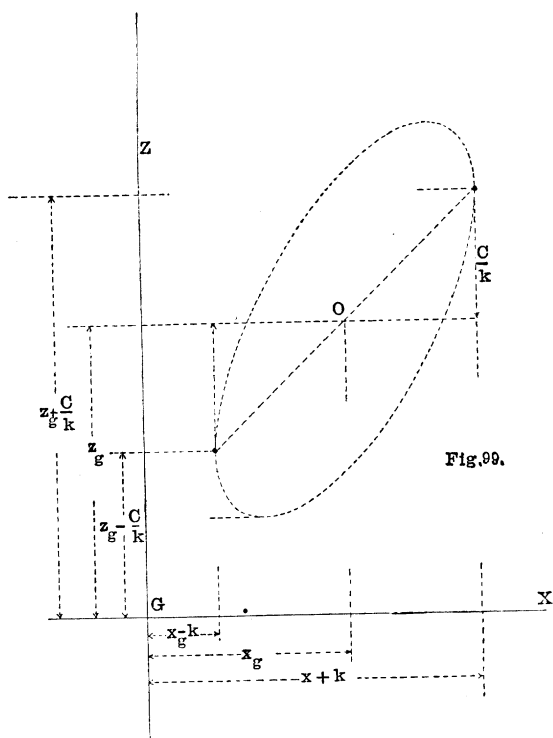
$$\begin{aligned} k : \frac{C}{\bar{k}} &:: \frac{k^2}{\bar{x}} : \frac{Ck^2}{\bar{k}^2\bar{x}} \\ &: \frac{C}{\bar{x}}. \end{aligned}$$

We have thus the lever arm

$$\bar{z} + \frac{C}{\bar{x}}$$

which has to be multiplied by $\bar{x} \cdot \Sigma m$, in order to give the product of inertia around G .

As to the multiplication of these three lengths by \bar{x} and \bar{z} , it is carried out in the second cord polygon as formerly.



iii. There are cases in which the above methods of finding the moments and products of inertia are not appropriate, and we give another method.

Referring to equations (1, 2, 3) and taking the central form,

$$\Sigma x^2 m = (\bar{x}^2 + k^2) \Sigma m \dots$$

we can, by simple algebraical decomposition, write them thus—

$$(\bar{x}^2 + k^2) \Sigma m = (\bar{x} + k) (\bar{x} + k) \frac{\Sigma m}{2} + (\bar{x} - k) (\bar{x} - k) \frac{\Sigma m}{2} \dots (4)$$

$$(\bar{x}\bar{z} + C) \Sigma m = (\bar{x} + k) \left(\bar{z} + \frac{C}{k} \right) \frac{\Sigma m}{2} + (\bar{x} - k) \left(\bar{z} + \frac{C}{k} \right) \frac{\Sigma m}{2} \dots (5)$$

iv. Equation (2) of this article can receive a generalization, which will be required when we come to treat of the Elastic Arch. It is this:

Let v be the coordinate axis from which x is measured at right angles to v , and u that from which z is measured at right angles to u , and calling the ordinates of O (figs. 97, 98) the central ordinates, we may express the generalization thus :

$$\begin{aligned}\Sigma xz &= \Sigma x \text{ ordinates from axis } v \times z \text{ ordinates from axis } u. \\ &= \text{Central ordinates } \bar{x} \text{ from } v \times \text{antipolar ordinate of } v \text{ from } u. \\ &= \text{Central ordinates } \bar{z} \text{ from } u \times \text{antipolar ordinate of } u \text{ from } v.\end{aligned}$$

For, the point F , fig. 97, determined by $\bar{z} + \frac{C}{\bar{x}}$ is the antipole of the axis of z (art. 117, 118, 128, 129) and is independent of the direction of the axis of x , and we can see from the construction that BO parallel to the axis of x can have any other direction u provided the ordinates from u are at right angles to it.

132. *Employment of the Methods of Last Article, to find the Moments and Product of Inertia of an Unsymmetrical Angle Iron around Axes going through Centre of Gravity of the whole.*—Fig. 100 is the cross section at an angle-iron, divided into three separate laminæ, 1, 2, 3, for each of which the central ellipse of inertia has been constructed, and then the moments and products of inertia around GX , GZ , and G are found, G being the centre of gravity of the whole lamina.

i. First of all, will be observed the construction of the moments

$$\Sigma mz : ab \text{ and } \Sigma mx : ab,$$

by means of the first force and cord polygons.

ii. Then in lamina 1 we require for it around the GX axis

$$\Sigma_1 mz^2 = (\bar{z}^2 + k^2) \Sigma m = (\bar{z}^2 + k^2) A_1 = \left(\bar{z} + \frac{k^2}{\bar{z}} \right) A_1,$$

of course : ab , or as it has been constructed : $a \cdot \frac{1}{2} \frac{A}{a}$.

Now, the intercept marked 1 on the line of weights of the second force polygon

$$= \Sigma mz : ab = A_1 \bar{z} : ab,$$

and comparing the construction of $\bar{z} + \frac{k^2}{\bar{z}}$ on the central ellipse of the 1 lamina, with that of the same on fig. 98, we have the lever-arm

$$\bar{z} + \frac{k^2}{\bar{z}}$$

with which to construct the intercept on the X axis in the first of the three second cord polygons, which consequently measures

$$A_1 \cdot \bar{z}_1 \left(\bar{z}_1 + \frac{k^2}{\bar{z}_1} \right) : abc.$$

iii. For the moments of inertia Σmx^2 around the GZ axis, compare in lamina 1 with fig. 97, and observe the intercept 1 on the second force polygon having its line of weights on the GZ axis, having the lever-arm

$$\bar{x} + \frac{k^2}{\bar{x}}$$

given to it for the intercept on the Z axis in second of the three second cord polygons.

The same operations are carried out for lamina 2, which being an exact square, they become simplified from all conjugate diameters being at right angles to one another, and no marking has been given on the figure.

iv. Lamina 3, being so near to the two axes, more especially to the GZ axis, the method of art. 131, iii. has been employed. For to carry out this method A_3 has to be halved on the line of weights of the first force polygon and carried into the construction of the first cord polygon.

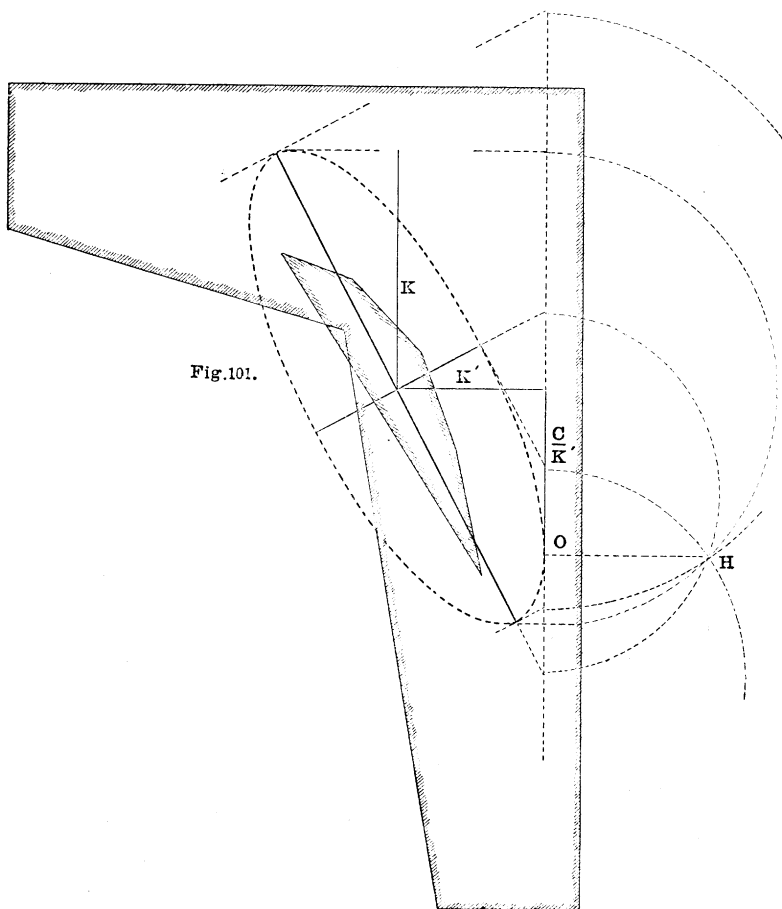
v. Note the construction of $\frac{C}{K'}$, comparing the construction in lamina 1 with fig. 98, and as a reason for taking K' compare with fig. 93. Note that $2K'$ has been taken as the pole distance

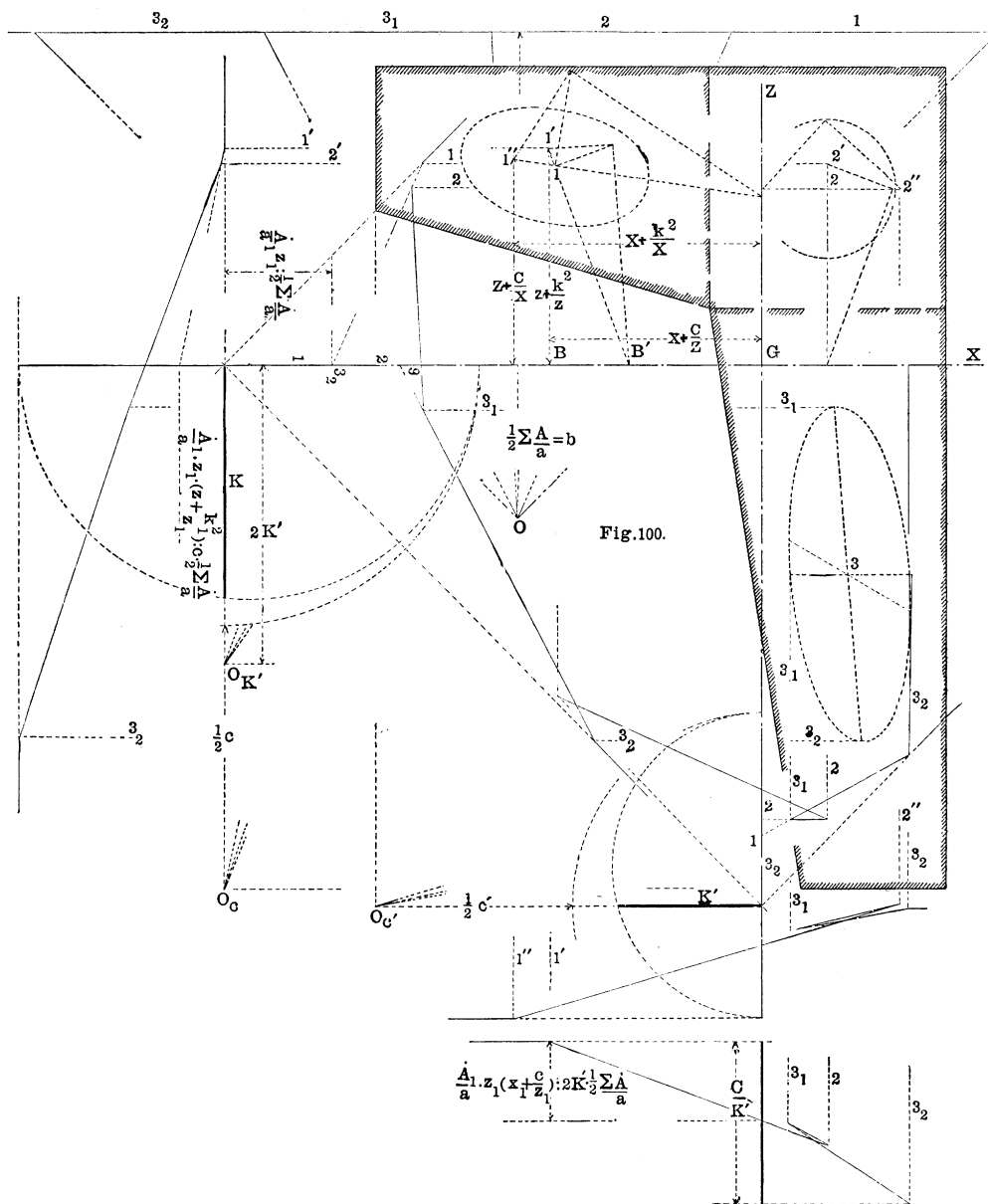
in the force polygon having Σmz for its line of weights where for

$$\Sigma mz \text{ is } A_1 \bar{z}_1 : ab, \text{ where } b = \frac{1}{2} \frac{A}{a}$$

and by taking $2K'$ we eliminate the $\frac{1}{2}$ multiplier in the b value.

vi. Having thus found K , K' and $\frac{C}{K'}$ for the axes GX , GZ , we are able after 127 (fig. 93) to construct the central ellipse of the whole lamina, and thence the core.





We have given these on another figure (fig. 101), as it would have confused the previous figure.

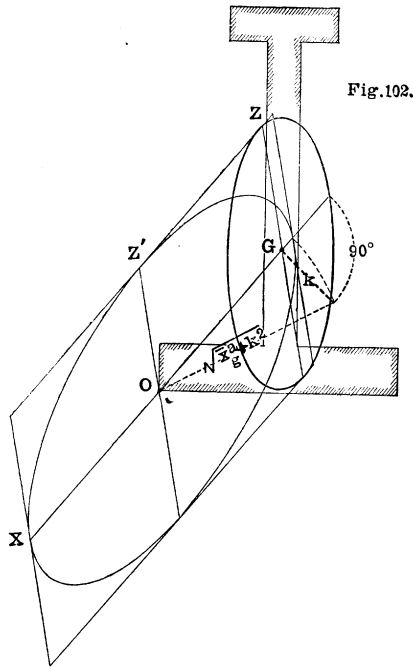


Fig. 102.

133. *Construction of Ellipse of Inertia around any Point of a Lamina, having given the Central Ellipse.*—Let O (fig. 102) be the point around which the ellipse of inertia is required. Unite the point O with the centre of gravity G of the lamina, by a line for the axis of X , and giving a diameter OG of the central ellipse. Through O , again, draw the axis OZ' parallel to the conjugate diameter of OG . Then in equation (2, 131)

$$\sum xzm = (\tilde{x}\tilde{z} + \tilde{C})\Sigma m,$$

and the diameters being conjugate, $\tilde{C} = 0$. Further $\tilde{z} = 0$, whence

$$\sum xzm = 0,$$

whence the axes chosen OX , OZ' are conjugate axes for the ellipse at O .

Further

$$b^2 = \frac{1}{\Sigma m} \cdot \Sigma z^2 m = \bar{z} + h^2 = h^2$$

(127, fig. 93); from this follows that the Z diameters in both the ellipses are equally long, *i.e.* $OZ' = GZ$.

Lastly

$$a^2 = \frac{1}{\Sigma m} \cdot \Sigma x^2 m = \bar{x}^2 + k^2,$$

whence the construction in fig. 102, which requires no further elucidation.

$\sqrt{\bar{x}^2 + k^2}$ has been wrongly marked on fig. 102, $\sqrt{\bar{x}^2 + k^2}$.

Section VI.—Construction of the Central Ellipse and Core of several Laminae, having Elementary Plane Forms.

134. *Central Ellipse and Core of a Parallelogram.*—In any figure it is well to find from inspection, if possible, the directions in which conjugate diameters exist, *i.e.* the directions around which we can say beforehand that

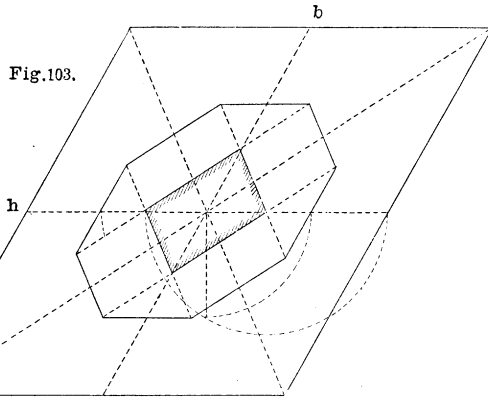
$$\Sigma xz = 0.$$

In this problem we see at once that the diameters of the parallelogram are conjugate diameters, that lines parallel to the sides, going through the centre of the figure, are likewise conjugate diameters, for in both cases it is evident that

$$\Sigma xz = 0.$$

Let h be the height of the parallelogram, b its breadth, then

$$\begin{aligned} k^2 \Sigma m &= \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} bz^2 \cdot dz, \\ k^2 &= \frac{b \cdot \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} z^2 dz}{b \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} dz} = \frac{\frac{1}{3} \left(\frac{h}{2}\right)^3 - \frac{1}{3} \left(-\frac{h}{2}\right)^3}{h} = \frac{\frac{1}{3} \cdot \frac{2h^3}{8}}{h} = \frac{h^2}{12} \\ &= (0.2887h)^2. \\ \therefore k &= 0.2887 \cdot h. \end{aligned}$$



In like manner

$$k' = 0.2887 \cdot b.$$

We can, with these values, lay down the ends of the conjugate diameters parallel to the sides of the parallelogram, from whence the ellipse may be constructed.

$$k^2 \Sigma m = \frac{h^2}{12} \Sigma m = \frac{bh^3}{12} \text{ and}$$

$$k'^2 \Sigma m = \frac{hb^3}{12}.$$

We might from these four points of the ellipse construct the core, as in figure, seeing they lie upon conjugate diameters, but we will also establish its form by simply substituting $\frac{h}{2}$ for \bar{z} and $\frac{h^2}{12}$ for k^2 in the expression $\bar{z} + \frac{k^2}{\bar{z}}$

$$\bar{z} + \frac{k^2}{\bar{z}} = \frac{1}{2}h + \frac{\frac{1}{2}h^2}{\frac{1}{2}h} = \frac{h}{2} + \frac{h}{2}.$$

The half height of the core is hence $\frac{h}{6}$, one-sixth the height of the parallelogram and thus occupies its central third.

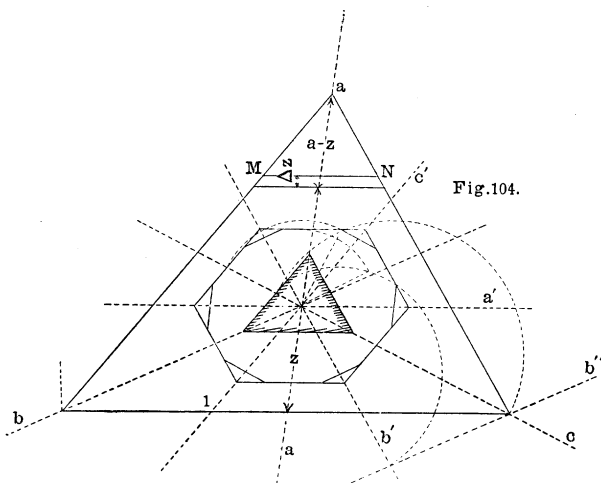
The core being known, we can proceed to obtain the length of the semi-conjugate diameters upon the diagonals of the parallelogram by means of the converse of the construction of (art. 130, fig. 95).

We can also obtain them as a simple fraction of the diagonal d . As the neutral axis pivots round the angle of the parallelogram, the circumference of the core describes a line. This line becomes the diagonal of a parallelogram similar to that given, but whose sides are $\frac{1}{6}$ th of it, its diagonals are likewise $\frac{1}{6}$ th of the diagonals of that given. This diagonal therefore cuts the diagonal d of the figure at a distance from its centre, $= \frac{1}{12}d$ and consequently

$$\frac{k^2}{\frac{1}{2}d} = \frac{1}{12}d \text{ whence } k = \sqrt{\frac{1}{24}d^2} = 0.204d.$$

We have thus eight tangents to the ellipse.

135. *Central Ellipse and Core of a Triangle* (fig. 104).—We observe beforehand that the line a which unites a vertex with the middle point of the opposite side l is conjugate with the line a' going through the centre of gravity and parallel to l , for this gives $\Sigma xz = 0$.



Of these conjugate diameters there are three pairs, a and a' , b and b' , c and c' .

Let us take the two conjugate diameters a and a' for our analysis. Let the line a be the axis of z and the point (a, l) the origin. Let $MN \cdot dz$ be an elementary area, then

$$MN : l :: a - z : a$$

or $MN = l \cdot \frac{a-z}{a}$ and $MN \cdot dz = l \cdot \frac{a-z}{a} \cdot dz$

the expression for an elementary area.

The moment of inertia of the elementary area around l as axis is

$$l \cdot \frac{a-z}{a} \cdot z^2 \cdot dz,$$

whence the moment of inertia of the whole triangle around 1 is

$$\int_0^a l \cdot \frac{a-z}{a} \cdot z^2 \cdot dz = l \cdot \frac{a^3}{12}.$$

In order to obtain k , refer to equation (4, 107) where

$$k^2 = \frac{\Sigma m z^2}{\Sigma m} - \frac{\bar{z}^2 \Sigma m}{\Sigma m}$$

and substituting

$$\frac{l \cdot \frac{a^3}{12}}{l \cdot \frac{a}{2}} = \frac{a^2}{6} \text{ for } \frac{\Sigma m z^2}{\Sigma m}, \text{ and } \left(\frac{a}{3}\right)^2 \text{ for } \bar{z}^2 = \frac{a^2}{9},$$

we have

$$k^2 = \frac{a^2}{6} - \frac{a^2}{9} = \frac{a^2}{18} = (0.2357a)^2.$$

Take again the expression

$$\Sigma m z^2 = \bar{z}^2 \Sigma m + \Sigma m z'^2$$

transposing

$$\Sigma m z^2 - \bar{z}^2 \Sigma m = \Sigma m z'^2.$$

For $\Sigma m z^2$ substitute $\frac{l a^3}{12}$.

For \bar{z}^2 „ $\frac{a^2}{9}$

$$\begin{aligned} l \cdot \int_0^a (l - \frac{z}{a}) z^2 dz &= l \int_0^a z^2 dz - \frac{l}{a} \int_0^a z^3 \cdot dz \\ &= l \frac{a^3}{3} - \frac{l}{a} \cdot \frac{a^4}{4} = l \left(\frac{a^3}{3} - \frac{a^3}{4} \right) = l \frac{a^3}{12}. \end{aligned}$$

For Σm substitute $\frac{la}{2}$.
we have

$$\frac{la^3}{12} - \frac{a^2}{9} \cdot \frac{la}{2} = \Sigma m z'^2 = \frac{la^3}{36}.$$

This then is the expression for the moment of inertia of a triangle around an axis a' , through its centre of gravity, and parallel to its base l .

In order to obtain the distance of the core from the centre of gravity, we have

$$\frac{k^2}{z} = \frac{\frac{a^2}{18}}{\frac{a}{3}} = \frac{a}{6}.$$

Lay off this distance as the antipole of the vertex in the line a , and draw through the point thus found a line parallel to l . Its completion requires no elucidation.

Additional tangents to the ellipse are found from the intersections of the core with the given conjugate diameters. The construction of two of these tangents is shown.

136. *Central Ellipse and Core of a Trapezium*, fig. 105.—We again observe beforehand, that the line c joining the middle points of the parallel sides a and b , is conjugate with the line c' going through the centre of gravity and parallel to these sides.

Divide the trapezium by means of a diagonal into two triangles whose contents are ah and bh , h being the height of the trapezium. The centres of gravity of both of these triangles lies in the third of h , and the distances of these centres of gravity from the centre of gravity of the trapezium are in the inverse ratio of a and b . Denoting the distance of the centre of gravity G of the trapezium from that of the triangle a by x , and from that of the triangle b by y , we easily obtain

$$x = \frac{b}{a+b} \cdot \frac{h}{3}, \quad y = \frac{a}{a+b} \cdot \frac{h}{3} \quad ^1$$

¹ From equality of moments of the two triangles around their common centre of gravity, we have $ax = by$, we have also $x + y = \frac{h}{3}$ whence are obtained the expressions in the text.

Finding now an expression for the moment of inertia of one of those triangles around the conjugate diameter c' . We have (135) the moment of inertia of a triangle around an axis through its centre of gravity and parallel to its base $2a$ with height h ; by substituting $2a$ for l and h^3 for a^3 , we obtain

$$\frac{2ah^3}{36} = \frac{ah^3}{18}.$$

Then in the general equation

$$\Sigma m z^2 = \bar{z}^2 \Sigma m - \Sigma m z'^2.$$

For \bar{z}^2 substitute $\left(\frac{b}{a+b} \cdot \frac{h}{3}\right)^2$.

For Σm „ ah .

For $\Sigma m z'^2$ „ $\frac{ah^3}{18}$,

and we obtain

$$\Sigma m z^2 = \left(\frac{b}{a+b} \cdot \frac{h}{3}\right)^2 \cdot ah - \frac{ah^3}{18}.$$

This can be put into the form

$$\left[\frac{1}{18} + \left(\frac{b}{3(a+b)}\right)^2\right] h^2 \cdot ah.$$

In a similar manner we obtain the moment of inertia of the b triangle around the same axis

$$\left[\frac{1}{18} + \left(\frac{a}{3(a+b)}\right)^2\right] h^2 \cdot bh.$$

Adding the two, we have the expression for the moment of inertia of the trapezium around c' , going through its centre of gravity

$$\left[\frac{1}{18} + \left(\frac{b}{3(a+b)}\right)^2\right] h^2 ah + \left[\frac{1}{18} + \left(\frac{a}{3(a+b)}\right)^2\right] h^2 \cdot bh.$$

If we divide this moment by the area $(a+b)h$ of the trapezium (a simple algebraical operation), we have

$$k^2 = \left(\frac{1}{18} + \frac{ab}{9(a+b)^2}\right) h^2.$$



In order to construct k , write it as follows

$$3k = \sqrt{\left\{ \frac{h^2}{2} + \frac{ab}{(a+b)^2} \cdot h^2 \right\}}.$$

Describe a semicircle, fig. 105, on the length h as diameter, then each of the sides A_1C of the inscribed isosceles triangle is $\sqrt{\frac{1}{2}h^2}$. The ordinate AB from the intersection of the two diagonals of the trapezium is

$$AB = \sqrt{\left\{ \frac{a}{a+b} \cdot h \cdot \frac{b}{a+b} \cdot h \right\}}.$$

Making $A_1B_1 = AB$, we obtain $B_1C = 3k$. One third of B_1C projected upon the middle line of the trapezium gives a semi-diameter k of the ellipse of inertia on c .

In order to obtain the semi-diameter k' upon c and around the axis c , complete the triangle of which the trapezium is a truncated part, then the semi-diameter k' is the difference between ak due to the whole triangle, and ak due to the added triangle.

The easiest method of obtaining this, is by transforming the trapezium into one correlated.

Halve a and b and join their middle points, and at right angles to this line and at these middle points lay off a and b and complete the trapezium the triangle of which it is a truncated part; we have a trapezium related to the last having the same value of k .

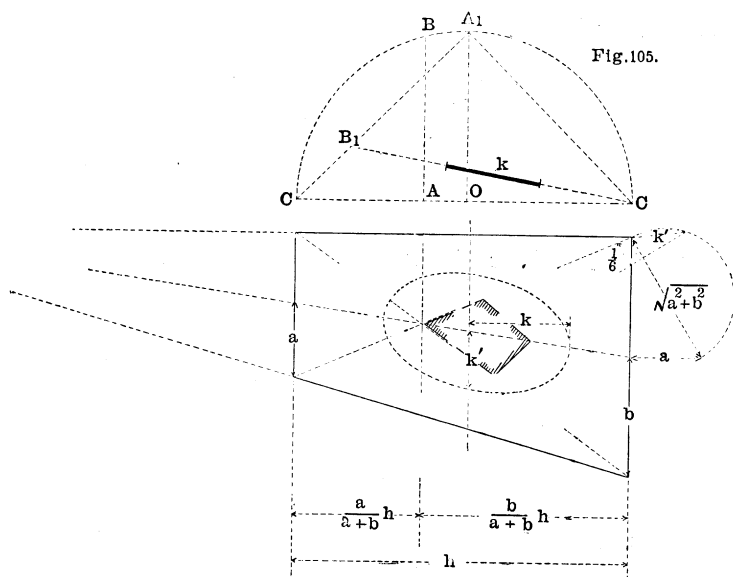
Let H be the height of the complete triangle, then

$$a : H - h :: b : H,$$

$$\text{or } H = \frac{b(H - h)}{a},$$

whence we obtain

$$H = \frac{bh}{a - b}.$$



| | Of the large triangle. | Of the small triangle. |
|--------------------------------|--------------------------------------|--------------------------------|
| Height | $\frac{bh}{b-a}$ | $\frac{ah}{b-a}$ |
| Contents | $\frac{b^2h}{b-a}$ | $\frac{a^2h}{b-a}$ |
| Moment of Inertia ¹ | $\frac{1}{6} \cdot \frac{b^4h}{b-a}$ | $\frac{1}{6} \frac{a^4h}{b-a}$ |

¹ This is derived from (art. 135).

For l substitute $\frac{bh}{b-a}$.

For $\frac{a^3}{12}$ " $\frac{b^3}{12} + \frac{b^3}{12} = \frac{b^3}{6}$,

and we obtain

$$\frac{1}{6} \frac{b^4h}{b-a}.$$

for

Consequently

$$k^2 = \frac{b^4 - a^4}{b^2 - a^2} = \frac{1}{6}(a^2 + b^2),$$

$$k = \sqrt{\frac{1}{6}(a^2 + b^2)},$$

whence the construction in the figure (*Euclid*, ii. 14, and i. 47).

Section VII.—Planes of Greatest and Least Stresses in a Beam.

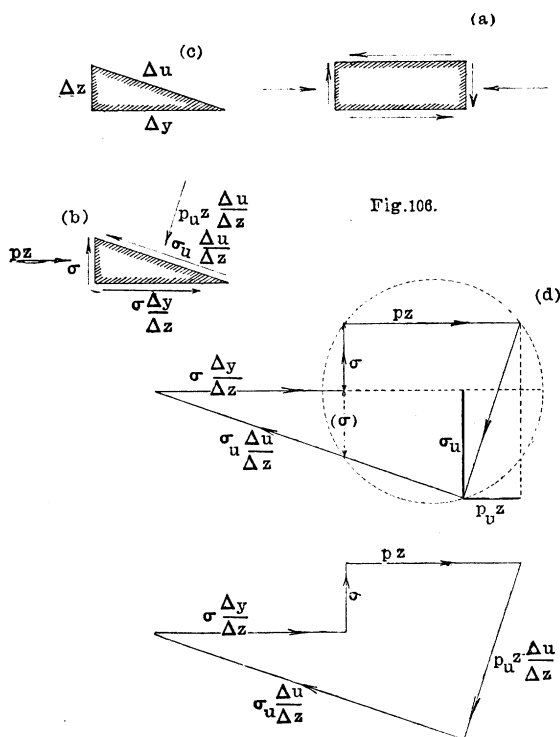


Fig. 106.

137. *Force Polygon of Stresses upon an Elementary Solid in a Beam.*—Returning to the two cross sections of a beam, separated Δy from each other. Consider the resistances p_z and σ units of bending and shearing respectively, at distance z from the neutral axis, then at the distance c of the extreme filament from the neutral axis through G , and on a line at right angles

to the vertical, lay off $\frac{pz}{2} \Sigma m$, and from its extremity draw a line through G . Any ordinate to this line from the vertical axis is $\frac{pz}{2} \cdot \Sigma m$, we have likewise given $\sigma \cdot \Sigma m$ for given values of z .

These two forces pz and σ act in planes parallel and perpendicular to the neutral plane (fig. 106a), and the questions arise, May not these strains be greater in some other planes? What are the stresses upon other planes? and, Upon what planes are the stresses greatest and least?

We will consider these in reference to the case where the neutral axis traverses the centre of gravity of the beam. Let the rectangle (fig. 106a) be an elementary solid of a beam, breadth unity, length Δy , viz. the distance between the two cross sections, we have thus six forces retaining it in equilibrium,

$$+ pz\Delta z, - p'z\Delta z, \sigma\Delta y, - \sigma'\Delta y, \sigma'\Delta u, - \sigma\Delta u,$$

as Δy diminishes p' approaches to p , so we may dispense with the accent.

Let it be cut diagonally by a plane Δu (fig. 106b), then we have five forces in equilibrium

$$pz\Delta z, \sigma\Delta z, \sigma\Delta y, \sigma\Delta u, pz\Delta u,$$

dividing all these forces by Δz , which does not affect the equilibrium, we have

$$pz, \sigma, \sigma \frac{\Delta y}{\Delta z}, \sigma \frac{\Delta u}{\Delta z}, pz \frac{\Delta u}{\Delta z}.$$

Being in equilibrium, they must give an inclosed force polygon.

In order to construct this force polygon, we have two forces pz and σ , given in magnitude and direction, the remaining three, in direction only, and the formation of the force polygon is accomplished by the following geometrical artifice.

Having drawn pz and σ (fig. 106d) extend the line of σ laying off on the extension $(\sigma) = \sigma$, and describe a circle around the three points thus given. From the end point of σ draw the



direction of the Δy plane, that is of the $\sigma \frac{\Delta y}{\Delta z}$ force. From the end point of (σ) draw that of the $\sigma \frac{\Delta u}{\Delta z}$ force, and from the other point where it cuts the circle draw a line parallel to the action of $pz \frac{\Delta u}{\Delta z}$, and the polygon is complete.

Elementary considerations prove that $\sigma \frac{\Delta u}{\Delta z}$ and $pz \frac{\Delta u}{\Delta z}$ are at right angles to each other.

Draw the two black lines σ_u and $p_u z$ from the end point in the circle of $\sigma \frac{\Delta u}{\Delta z}$, respectively parallel to σ and pz , and otherwise as in figure 106*d*). These are the forces per unit of area on the Δu plane.

1. In reference to σ_u . Compare the similar triangles formed by the elementary solid and that of which $\sigma \frac{\Delta u}{\Delta z}$ is the hypotenuse, and we have the proportion

$$\Delta u : \Delta z :: \sigma \frac{\Delta u}{\Delta z} : \sigma_u.$$

2. In reference to $p_u z$. Compare the triangle formed by the elementary solid and that of which $p_u z \frac{\Delta u}{\Delta z}$ is the hypotenuse

$$\Delta u : \Delta z :: p_u z \frac{\Delta u}{\Delta z} : p_u z.$$

138. *Planes of Maxima and Minima Stresses in the Elementary Solid.*—The construction of last article can be applied to the determination of the maxima and minima stresses and of the planes on which they act.

Let the trace

$$\sigma \frac{\Delta u}{\Delta z}$$

of the cutting plane turn round the end point of (σ), then (figs. 107*a, b, c, d*) show the maxima and minima of the stresses and the planes in which they act.

In figures 107, as the plane Δu (or u) turns round from its position in fig. 106*d* in the direction of the hands of a watch, it cuts (107*a*) the circle in the point A , σ_u is evidently then a maximum. Continuing to revolve it cuts (107*b*) the circle in the beginning of the next quadrant B where $p_u z$ is a maximum and σ_u zero, still revolving (107*c*) it cuts the circle in the beginning of the third quadrant C , σ_u is again a maximum, and when it has completed that quadrant (107*d*) and cuts in D , $p_u z$ is a minimum and σ_u is again zero.

139. *Curves of Maximum and Minimum Stresses laid down as Ordinates from the Vertical and Principal Axis of a Beam.*—

By taking the cross section of a beam, we can conveniently lay down curves, whose ordinates give the maxima stresses at any point under given shearing and bending stress. For example, taking the cross section of the rail (Plate I.), lay down the oblique line through G so as to be locus of all values of $\frac{1}{2} p z \cdot ab$

(b having been taken = $\frac{1}{2} \cdot \frac{A}{a}$, then

$$ab = a \cdot \frac{1}{2} \frac{A}{a} = \frac{1}{2} A),$$

and let us construct the ordinates of maximum stress for ΔA_6 , i.e. for lamina 6, we have here $\frac{1}{2} p z \cdot ab$ as ordinate from the vertical axis. Then at the end of that ordinate as centre and with the compasses spanning from thence to the extreme point of $\sigma_6 \cdot ab$ as radius, describe a semicircle on its horizontal diameter. This semicircle cuts this horizontal line in two points measuring from the vertical axis, the maximum and minimum stress $p_u z$ upon a plane u (whose position we will indicate further down).

This construction having been repeated for a number of points, we obtain the two lines marked maximum and minimum.

These lines give also the maximum shearing stress by measuring horizontally from the oblique line, for (138) the radius of the circle at any point is the measure of the maximum shearing stress at that point.

The force polygon thus constructed for ΔA_6 is shown but without the auxiliary semicircle at fig. *a*, plate I., in order to afford a clear comparison with figures (107*b* and *d*).

140. *Tractories traced upon the Longitudinal Section of the Beam, giving the Direction of the u Plane for any Point.*—The angle δ_6 upon the figure of the rail is the angle which the plane u makes with the horizontal at 6 in the longitudinal section, and it is evident that it alters its value for every value of z . If then we suppose it to keep the same direction for an indefinitely short length Δu , δ then alters its value, and if we carried out this gradual alteration we would obtain a line belonging to the class of tractories, the trace on the longitudinal section of a plane in a beam, along which the maximum and minimum of bending stress occur, but on which shearing stress is zero.

Comparing (figs. 107*a*, *b*, *c*, *d*) with each other, we note the leading characteristics of the planes u .

i. The planes of maxima at any point make angles of 45° with each other. Notice the position of the plane u in (fig. *a*) where σ_u is a maximum, keeping in view that

$$p_u z \frac{\Delta u}{\Delta z}$$

and σ_u or u are always at right angles to each other, notice that while u turns from *A* to *B* one extremity of

$$p_u z \frac{\Delta u}{\Delta z}$$

passes from *A* to *B*, and its other extremity being fixed on the circumference of the auxiliary circle, its direction has passed through 45° (*Euc.* iii. 20),

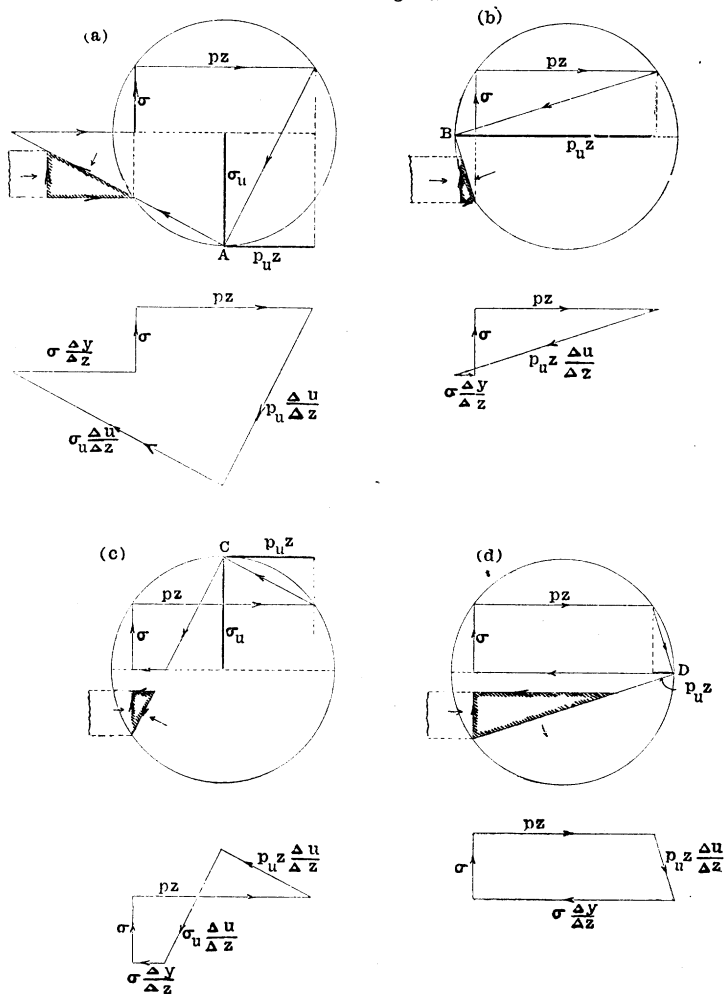
$$p_u z \cdot \frac{\Delta u}{\Delta z}$$

thus travels through 45° as its revolving extremity passes from *A* to *B*, from *B* to *C*, from *C* to *D*.

ii. $p_u z$ is of the same sign as $p z$.

iii. At the neutral axis n , $p z$ being zero, the traces of the planes u , which give the maximum and minimum, *i.e.* the

Fig. 107.



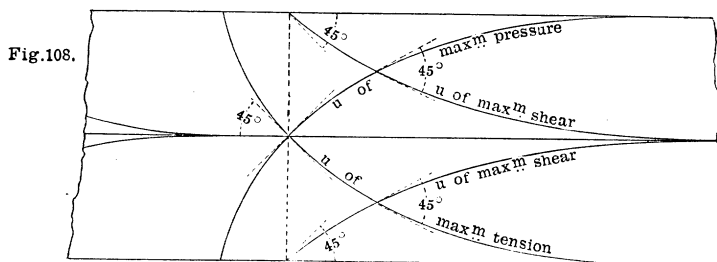
maximum of opposite signs of stress cut the axis at angles of 45° .

iv. At the neutral axis the plane u of maximum shearing stress is parallel to the axis.

v. At the outside laminæ the planes of maximum bending stress are parallel to the axis.

vi. At the outside laminae the planes of maximum shearing stress make angles of 45° with the planes of maximum bending stress.

Fig. 108 is a representation of the direction of these u planes upon a beam of uniform breadth, so as to give at a view the characteristics just mentioned.



These tractories have a direct practical value in the scarfing of compound wooden beams in countries where wood is abundant and iron dear, for the corresponding teeth of two beams ought to cross the plane of maximum shearing at right angles and follow the lines of no shearing.

141. *Construction of the Lines u of Maxima Stresses on a Beam.*—Let fig. 109 be the cross section, and fig. 110 (plate III.) the half longitudinal section of a beam supported at the two ends, and loaded uniformly with *one ton per lineal foot*.

For the cross section reducing base

$$a = 1'', b = 10'', c = 10''$$

also half-height of beam = $10''$, then

$$pc \cdot ab = pc \cdot 10.$$

As the beam is symmetrical above and below the neutral axis through G we will carry out the construction for the moment of inertia upon one half of the cross section of the beam giving us

$$\frac{1}{2} \frac{\sum mz^2}{abc}$$

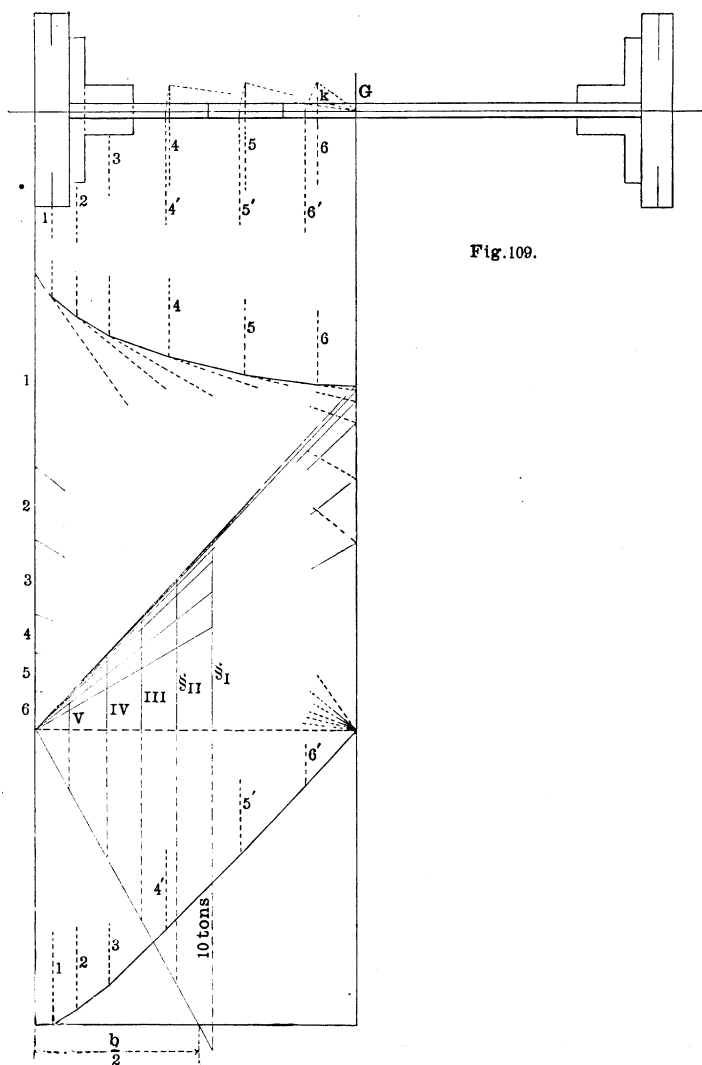


Fig.109.

and we have

$$p \cdot \Sigma m z^2 = Z y = \text{bending moment}$$

$$\text{or} \quad \frac{1}{2} p c \cdot ab \cdot \frac{\Sigma m z^2}{abc} = \frac{Z y}{2}$$

but

$$\frac{\Sigma m z^2}{abc} = \text{full intercept of second moment cord polygon}$$

whence

$$\frac{1}{2} p c \cdot ab \times \text{intercept of second moments} = \frac{Z y}{2}$$

Now the intercept of the cord polygon of the longitudinal section gives

$$\frac{Z y}{h}$$

and in order therefore that we may obtain

$$\frac{1}{2} p c \cdot ab$$

as the bending moment intercept of the cord polygon we must have

$$\frac{1}{2} p c \cdot ab \cdot \frac{2 \Sigma m z^2}{abc} = Z y$$

and make

$$\frac{2 \Sigma m z^2}{abc} = \text{the pole distance } h,$$

that is, if cross sections and longitudinal sections are drawn to the same scale.

In our example the longitudinal section is to a scale of one half that of the cross sections, and as the bending moment ordinates are inversely as the pole distance, we must take

$$h = \frac{1}{2} \left(\frac{2 \Sigma m z^2}{abc} \right)$$

in order to obtain $\frac{1}{2} p c \cdot ab$ as the ordinate.

Returning to cross section fig. 109, let us lay off $Z = 10$ tons, as in fig. 85 and plate I., with this difference, that as we have

$$\frac{1}{2} \cdot \frac{\Sigma m z^2}{abc}$$

only, we lay off not b but $\frac{1}{2}b$. We have now on the line marked S_1 (fig. 109) the values of $S.b$ for six values of z on either side of the neutral axis for the section under the front of support at which there is no bending moment, and at S_2, S_3, S_4, S_5, S_6 (figs. 109, 110) the shearing stress decreases till at S_6 , the centre of the beam, there is no shearing and the bending moment is greatest.

For S_1 (fig. 111), the angle δ is constant and equal 45° for all values of z , for the centre of the auxiliary circle is always upon the vertical axis.

For S_2 (fig. 112) the section which the student will find most easy to follow, having laid off from the longitudinal section the corresponding value of

$$\frac{1}{2}pc . ab$$

and the appropriate curve of shearing, viz., the values of Sb (compare fig. 85), repeat the construction shown on the figure of the rail for ΔA_6 (Plate I.) so as to obtain the curve of maximum tensions and pressures, noting that at certain values of z there are two values of x and therefore at these points requiring two constructions, the greater value of x giving the lesser value of stresses per unit of area, and a smaller angle δ .

Transfer these angles, for facility of reference, to a convenient part of the paper, and find their reflexions (fig. 112a). The reflexions are the values of δ for the other half of the beam.

Having made these constructions for all the sections, transfer the little crosses to their appropriate places in the longitudinal section. Interpolate them, as shown on part of the section, by some simple geometrical artifice, and through them trace as many tractories as we please.

In this example the tractories at the angle-irons and tables of the girder become in the one case so flat as to be scarcely distinguishable from the horizontal, crossed by the others scarcely distinguishable from the vertical, so we have only drawn in tractories upon the web.

If we laid off lines crossing these tractories at angles of 45° , we would have the lines of maximum shearing.

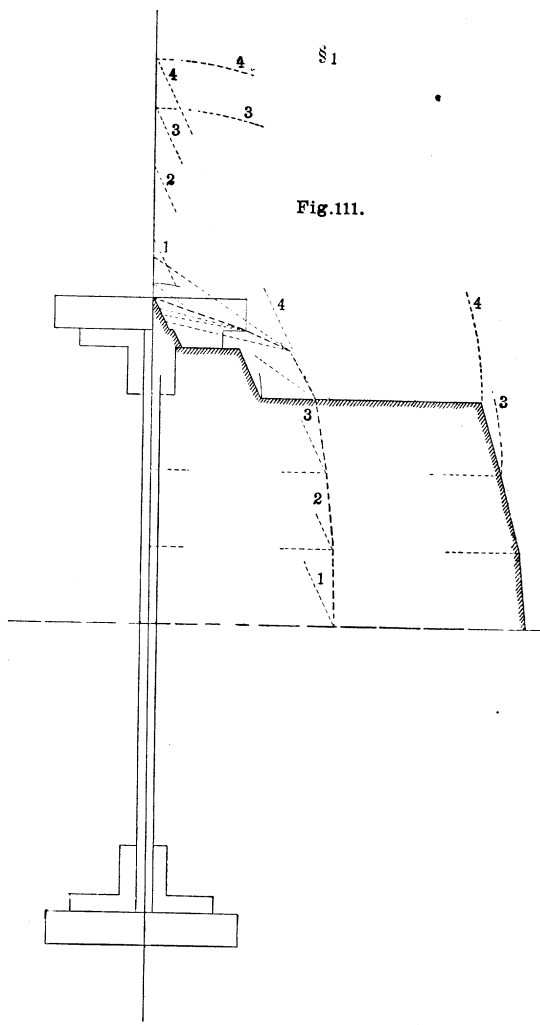


Fig.111.

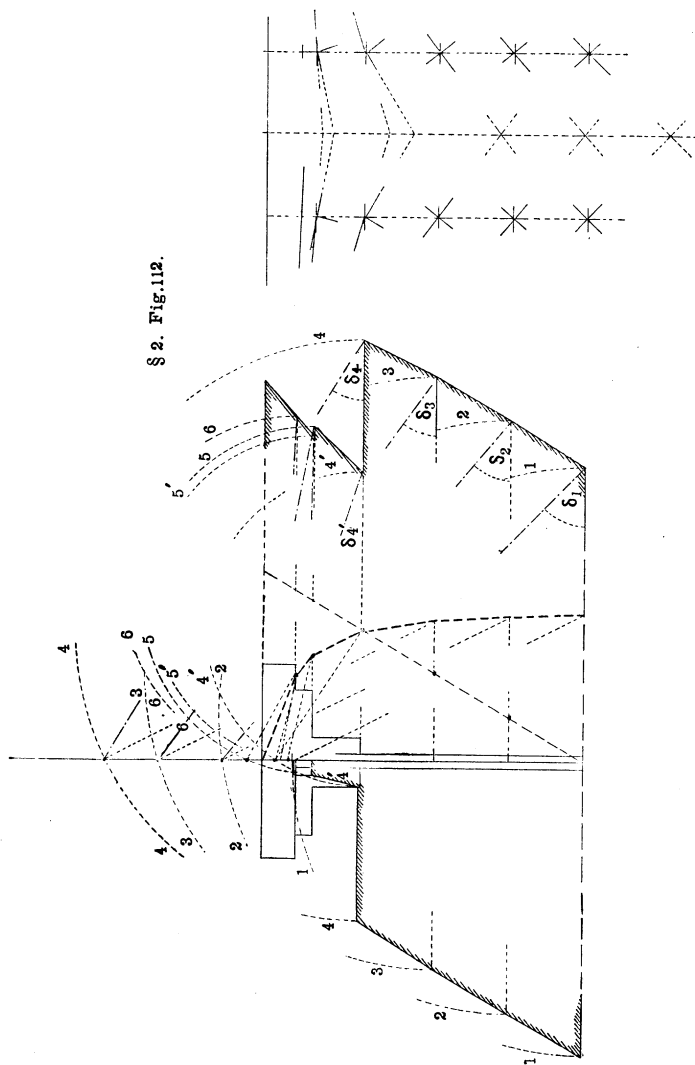


Fig. 112.

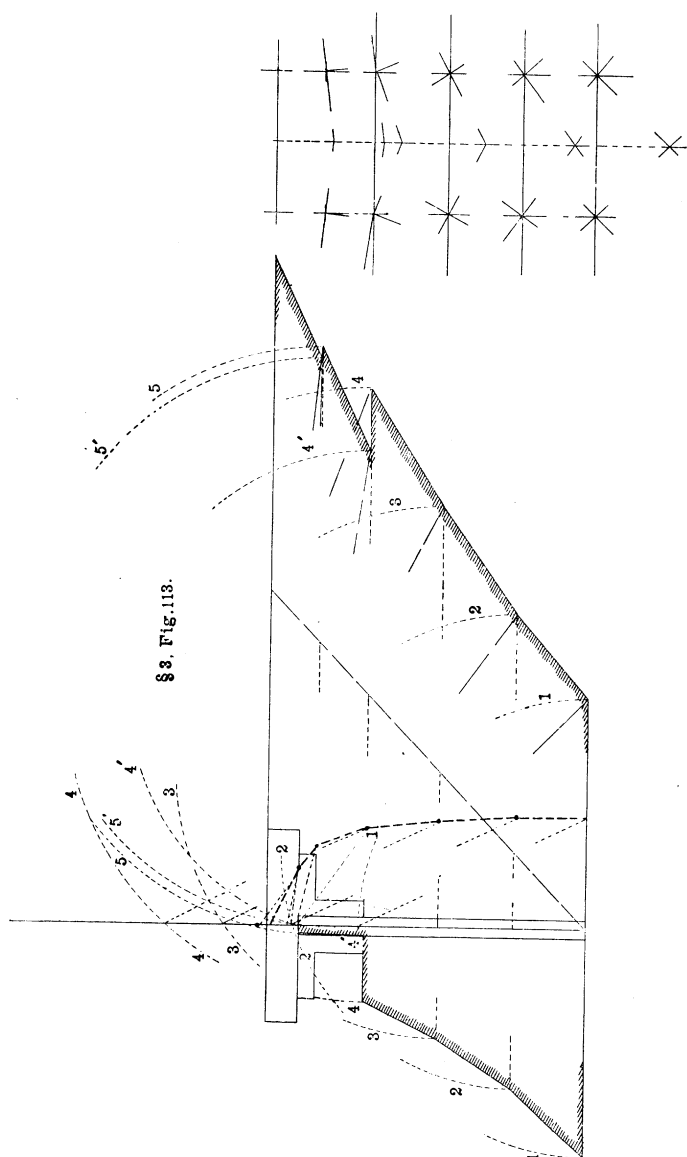
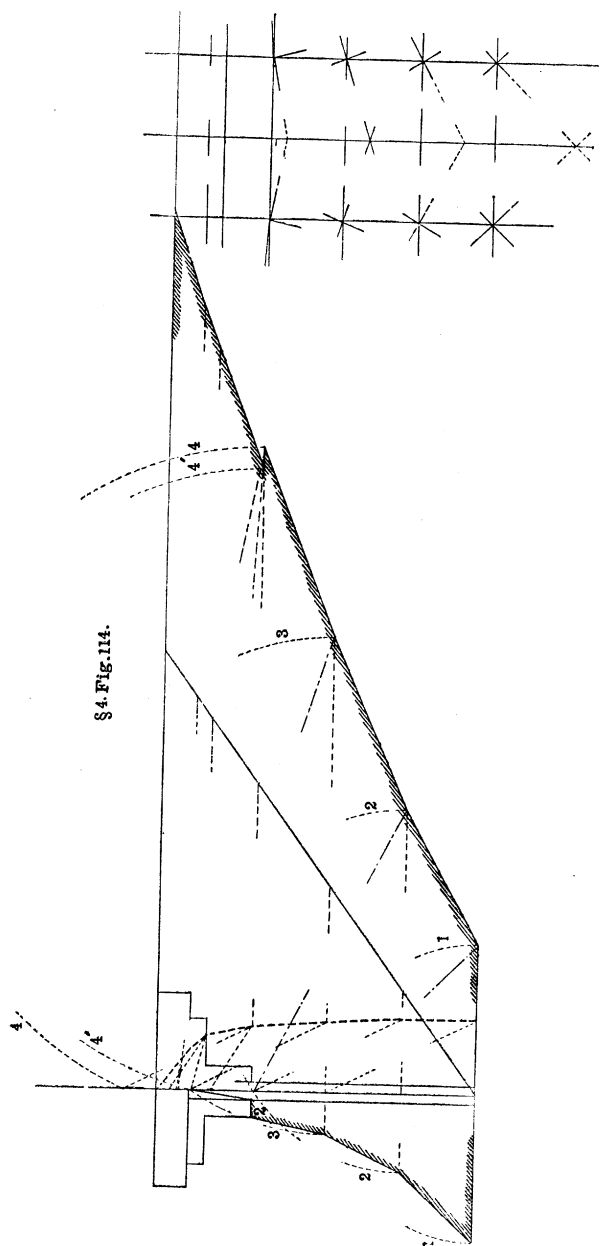
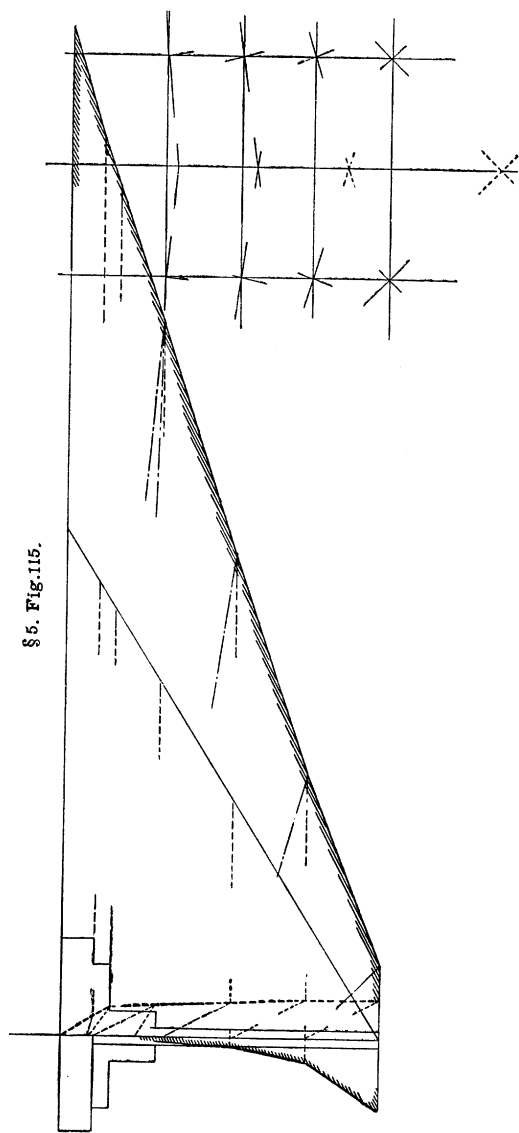


Fig. 113.





CHAPTER IV.

CONTINUOUS BEAM.

142. *Preliminary Remarks on Measure of Bending Moments arising from Forces applied Outside of Points of Support.*

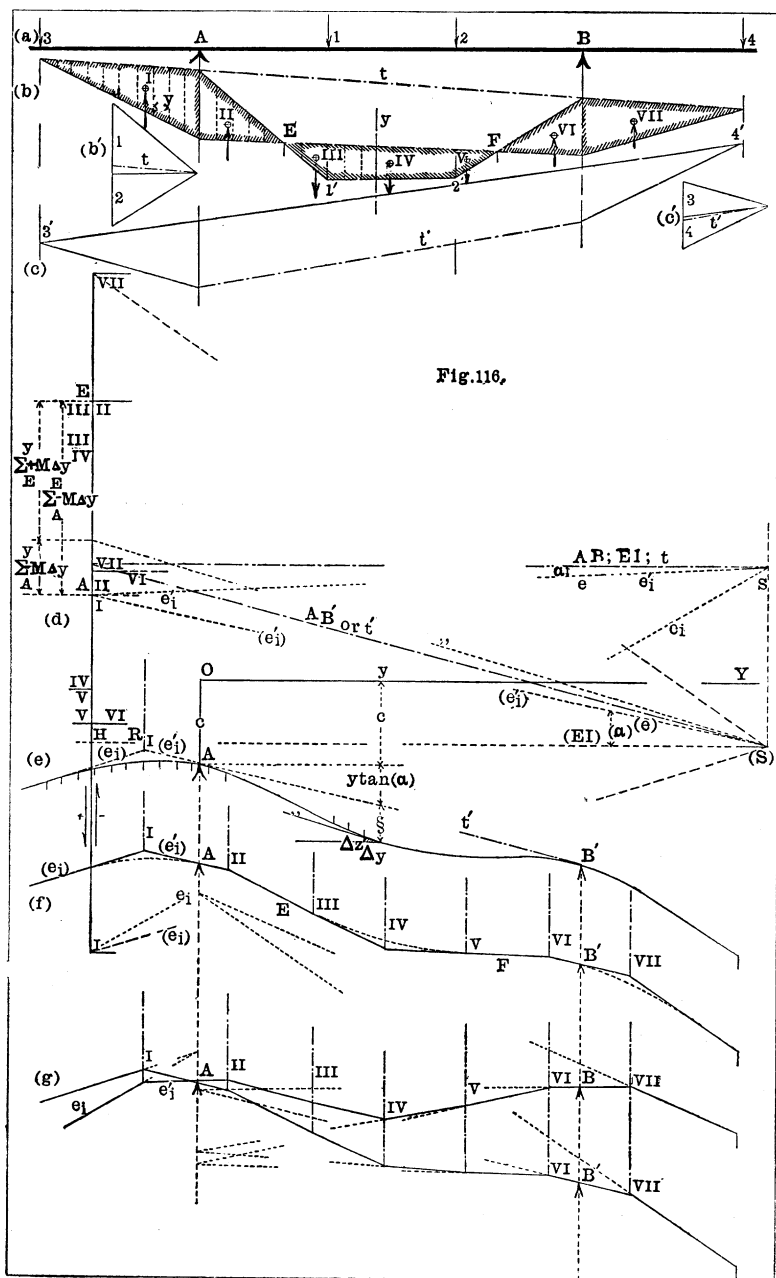
i. Consider a beam, AB (fig. 116a), overhanging its two supports, A, B . Let it be first loaded only with weights, 1, 2, between the supports, and from its force and cord polygons (fig. 118b, b'), and $A\ 1'\ 2'\ B$, any ordinate of the latter (31) measures the bending moment at that point: pole distance h , and the beam is bent downwards.

Let it be now loaded only with weights 3, 4 beyond its supports and form its force and cord polygons with the same pole distance (fig. 116c, c'); then again any ordinate of the latter measures the bending moment at that point. Pole distance h , and the beam is bent upward.

It will be well here formally to recognise that in the case of bending moments arising from forces exterior to a span, the ordinates of the bending moments within the span are bounded at both extremities by two straight lines.

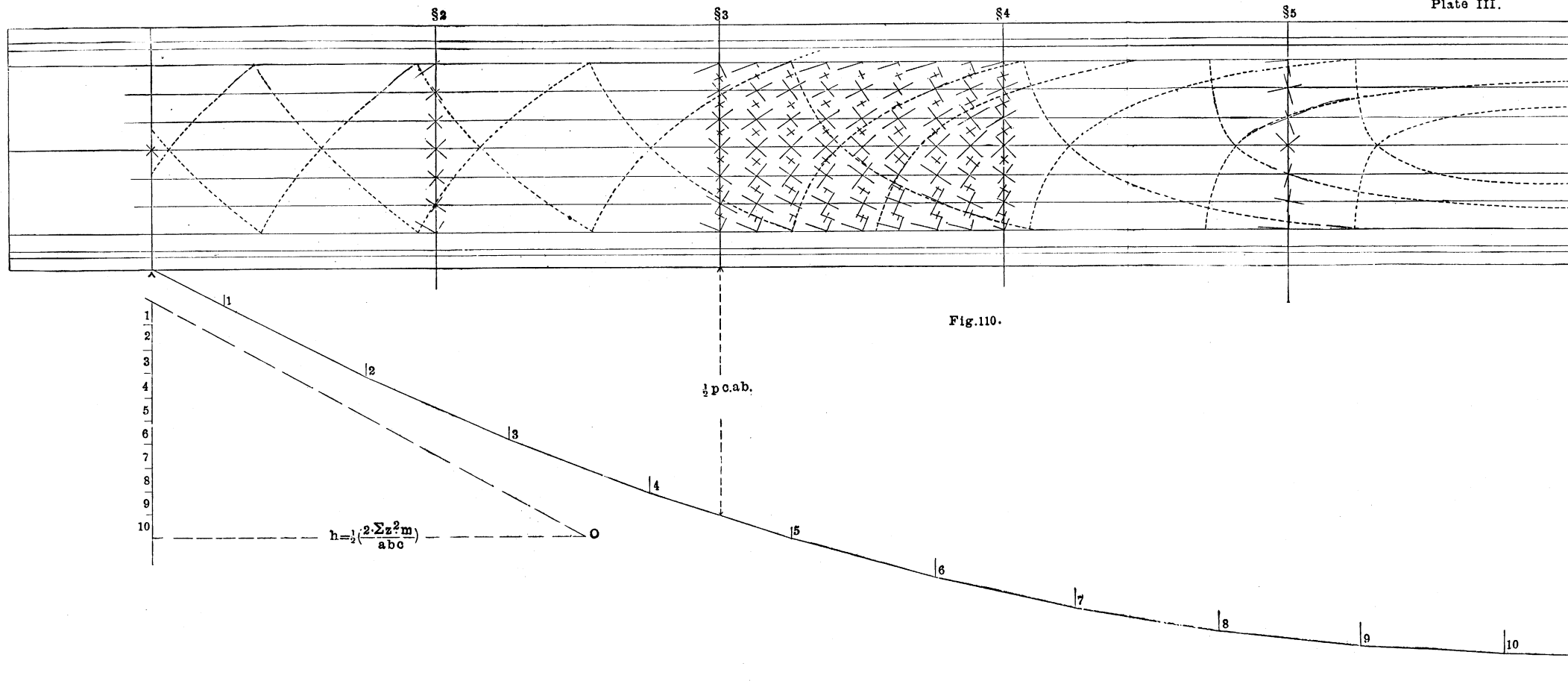
Let the beam now be loaded with the above weights, both within the span and exterior to it, then the forces bending downward and bending upward strive for the mastery over one another. And the algebraical sum of the ordinates of the two bending moment or cord polygons, at any point (38) measures the whole bending moment at that point: pole distance.

Applying then the two cord polygons to one another, so that the line $3'\ 4'$ of the negative moment polygon (fig. 116c), may coincide with the closing line t of the positive moment polygon, and the verticals through the supports be the same, any ordinate within the shaded spaces (fig. 116b) measures the whole bending moment of the beam at that point. Where the ^{positive} negative ordinates are greatest the beam is bent ^{downward} _{upward}, where the



To face page 196.

Plate III.



ordinates as at E, F are equal to each other, the sum is zero, and the beam there remains straight. These two points, E, F , are the points of inflexion of the beam under the given weights.

143. *Professor MOHR'S Problem.*¹ *Geometrical Interpretation of the Equation to the Elastic Line.*—In art. 101, equa. 12, we have found the equation which connects the curvature, assumed, by the neutral axis of a beam under a load with its bending moment to be

$$M = \frac{EI}{r}.$$

We deduce from this equation that according as M is positive, negative, or zero, r is positive, negative, or infinite, and the neutral axis of the beam is curved downwards, upwards, or straight.

The form thus assumed by the neutral axis is called the elastic line.

The differential equation of this line (art. 101, equa. 15) is

$$\frac{d^2 z}{dy^2} = \pm \frac{1}{EI} \cdot M \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating once

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{EI} \cdot \int M dy \quad . \quad . \quad (2)$$

If then we take the area of the moment ordinates of a beam (as fig. 116*b*) and divide it into vertical layers Δy broad, each layer will have an area

$$\frac{M}{h} \cdot \Delta y.$$

To get rid of constants unnecessary to our present purpose, let us for a short time consider

$$h = 1,$$

then area of layer is

$$M \Delta y.$$

¹ Beitrag zur Theorie der Holz-und Eisenconstruktionen, Zeitschrift des hannöv, Arch. und Ing.-Vereins, 1868.

Consider now the beam loaded with the values $M\Delta y$ as forces, and as forming the line of weights of a new force polygon which we will refer to as the $M\Delta y$ force polygon. These areas being reduced to representative lines in the usual way by a base a , which for the same reason as above we will also consider as equal to unity.

In this figure (fig. 116*d*) the individual values of $M\Delta y$ have not been retained, but the moment area having been divided into well-defined portions, I., II., III., . . . these are shown on the line of weights. The negative values of $M\Delta y$ are laid upward successively and the positive downward, all being very distinctly marked. A pole distance named (EI) has been chosen giving a pole (S), the position of (S) being otherwise arbitrary.

Taking A as origin on the bending moment cord polygon (fig. 116*b*), and y any horizontal distance from A , then from the corresponding point A on the line of weights of the $M\Delta y$ force polygon (fig. 116*d*) the

$$\Sigma -M\Delta y$$

from A to E has been marked upwards, then the bending moments changing sign

$$\Sigma +M\Delta y$$

from E to y has been marked downwards, and the sum of these two

$$\Sigma -M\Delta y + \Sigma +M\Delta y = \Sigma_A^y M\Delta y$$

is marked, we hope, so clearly as to require no further explanation.

Recalling now (equa. 2)

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{EI} \int_0^y M dy,$$

or infinite differences

$$\frac{\Delta z}{\Delta y} \mp \text{constant} = \pm \frac{1}{EI} \Sigma_A^y M\Delta y$$

when

$$y = 0, \quad \frac{dz}{dy} = \tan(\alpha) = \frac{AH}{(EI)}$$

whence

$$\frac{dz}{dy} = \tan(\alpha) \pm \frac{1}{EI} \int M dy, \quad \dots \quad (4)$$

which, as above, can be translated into the symbols of finite differences.

Integrating again we obtain

$$z \mp \text{constant} = y \cdot \tan(\alpha) \pm \frac{1}{EI} \int_0^y dy \int_0^y M dy. \quad \dots \quad (5)$$

From the $M\Delta y$ force polygon form a cord polygon. Let OY be the horizontal axis of abscissæ

$$OA = c,$$

the ordinate over A , then the ray $(S)A$ is the corresponding ray to that Δy of the cord polygon which passes through the support A (fig. 116*e*), and at the distance y gives

$$y \tan(\alpha)$$

as part of the ordinate z . c is another part, and the remaining portion of the ordinate marked § evidently measures that part of equa. 5, viz.,

$$\frac{1}{EI} \int_0^y dy \int_0^y M dy$$

that is,

$$\frac{1}{EI} \int_0^y \Delta y \int_0^y M \Delta y = \frac{1}{EI} \int_0^y M \Delta y \cdot y,$$

the exterior intercept of the cord polygon (the $\Sigma_1^2 - Px : h$ of fig. 13, p. 21) whence this second cord polygon satisfies the equation to the elastic line (equa. 5), and is therefore an elastic line.

ii. *Construction of Tangents to the above Elastic Line.*—We have marked the position of the centres of gravity of the moment area portions *I, II, III. . . .* by a small circle.

Consider now area *I*. Then instead of constructing the elastic line cord polygon from the individual values of $M\Delta y$ we may take a group of them as *I*. (*I*—*I*. on the $M\Delta y$ force polygon) and draw in only the extreme rays (e_i) and (e_i') corresponding to (*S*) *I*. and (*S*) (*I, II*), cutting the vertical R_I going through the

centre of gravity of the group (*i.e.* of area I), these extreme rays are necessarily tangential to the cord polygon at the points 3 and A (fig. 17), and thus giving

$$\frac{\Delta z}{\Delta y} = \tan(a)$$

at A as formerly.

The same may be done with the other areas, II , III , (as in fig. 116*f*), and then we have a cord polygon necessarily tangential to the elastic line at the points 3, A , E (III, IV), (IV, V), F , B , 4.

This is all that is required of the elastic line. We shall call this latter form the tangential envelope to the elastic line, or sometimes, more shortly, tangential envelope.

iii. We have obtained *an* elastic line which lies upon one of the given supports A ; but the other support B' depends upon the position of (S), and in order that the elastic line may lie upon a given support B , we must apply Culmann's theorem that as the side $\overline{I, II}$ of the given tangential envelope (10, 52) cuts the side $\overline{I, II}$ of the required elastic envelope in A , all corresponding sides will cut each other in the vertical through A , whence extending the side $\overline{VI, VII}$ going through B' till it cuts the vertical through A in a point. From that point draw the required corresponding side through the given support B . A new tangential envelope passing through two given supports A and B , derived from that found passing through A and B' is shown in fig. 116*g*.

This is equivalent to removing the pole (S) to another position S in the same vertical.

As the supports A, B , in the new elastic line have been taken on a horizontal line in (fig. 116*g*), in (fig. 116*d*) AB now coincides with EI and (e_i) and (e'_i) take the new positions e_i and e'_i .

AB' is *an* elastic line in which

$$\tan(a) \text{ or } \frac{dz}{dy}$$

at support A is entirely arbitrary and determines the point of support B' and the

$$\tan (\beta) \quad \text{or} \quad \frac{dz}{dy}$$

at support B' .

But given the positions of A and B , $\tan \alpha$ is no longer arbitrary, for, joining AB by a right line t , that line is the closing line of the cord polygon called the elastic line, or that other called the tangential envelope, that is, the closing line to the cord polygon of all the $M\Delta y$ forces, acting between A and B , viz.

$$\Sigma_{II}^{VII} M\Delta y.$$

forces; $\tan \alpha$ and $\tan \beta$ are *thus dependent only on the moments within the span and over the points of support*.

iv. Again, $\tan \alpha$ and $\tan \beta$ would not be altered if we formed an elastic line or its tangential envelope with the *full* values of the negative and positive moment areas (fig. 116*b*), and supposing each of them concentrated in its centre of gravity.

For the pole S of the force polygon being the same, the point of support A being the same,

$$\Sigma_A^B M\Delta y$$

is the same, whence the closing line t must be the same. B is therefore the same, and the reactions at A and B measured on the line of weights from the closing line must be the same, the extreme rays e , e' the same; whence

$$EIe = \angle \alpha \quad \text{and} \quad EIe' = \angle \beta$$

just as before. *What we principally require of the elastic line are its tangential lines over the points of support*, which this method gives. It will be generally used in the sequel.

v. In our figure e'_i is also e as it is the extreme ray of the elastic line cord polygon at A , in fact the tangent line at A , e' is the tangent at B , and its correspondent on the force polygon is $S(VI, VII.)$, not drawn to avoid confusion. Whence (e'_i) being also e , it follows that the elastic line has the same line for tangent on both sides of a point of support. This is the important property of *continuity of the elastic line*.

vi. In the foregoing geometrical interpretation of the equation to the elastic line we have three constants, h , α , and EI , which

we here call $\tilde{e}i$, though not to crowd our notation with them we proceeded as if h and a were unity, so that in reality instead of

$$z = c + y \cdot \tan a + \frac{1}{EI} \int_0^y dy \int_0^y M dy$$

we have obtained

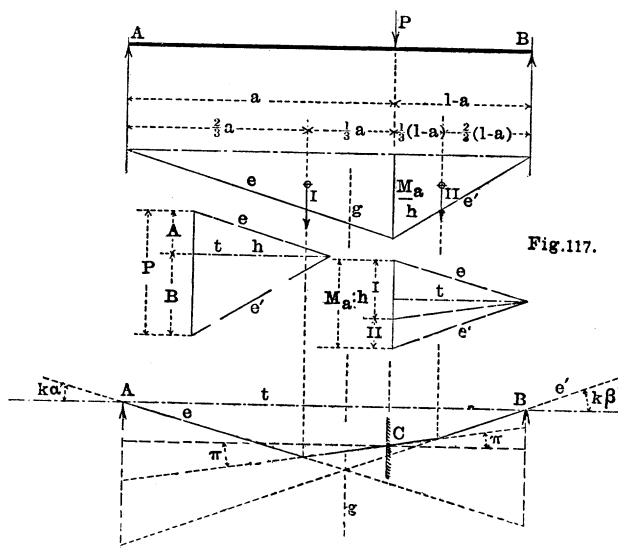
$$z = c + \frac{1}{h \cdot a \cdot \tilde{e}i} \cdot y \cdot \tan a + \frac{1}{h \cdot a \cdot \tilde{e}i} \int_0^y dy \int_0^y M dy,$$

whence the variable part of the ordinate z is

$$\frac{EI}{h \cdot a \cdot \tilde{e}i}$$

times larger than the ordinate of the real elastic line.

144. *Construction of Elastic Line Tangent Envelope of a Beam lying freely upon Two Supports A and B with forces only between them.*



(a) One load P upon the span at a point $y = a$ from origin at A .

The first or moment area cord polygon gives the moment area : h (31) forming a triangle, and if we reduce this moment area : h to a representative line in order to form the line of weights

of the second or Mdy force polygon by means of the base $\frac{l}{2}$ (15),

it is evident that the moment ordinate $\frac{M_a}{h}$ in the vertical of P represents the area of this triangle.

Let this reducing base be accepted, then $\frac{M_a}{h}$ is the length of the line of weights upon the $M\Delta y$ force polygon.

Finding then the centre of gravity of the moment area triangle, draw through it a vertical g . Taking now a pole distance $\frac{l}{3}$ (preferred for reasons which will afterwards appear) for the Mdy force polygon, and A for the point of support in the vertical A , draw through A the extreme ray e of the elastic line envelope cord polygon cutting g in a point. Through this point draw the other extreme ray e' cutting the vertical through B in some point B' . This gives a provisional elastic line. Join AB' for t' , which transfer to the force polygon, cutting the line of weights in the point (I, II). We can now, drawing t parallel to the given supports AB , shift the pole of the force polygon, or according to the method (134, iv.) in the general problem, find the true e, e' , giving

$$k \tan \alpha, k \tan \beta$$

for the tangents of the angles at AB . In our figure we have not shown a provisional elastic envelope.

This tangential envelope gives the two tangents over the points of support, but fig. 117 represents it carried out to three tangents at the points A, B, C , by taking the centres of gravity of the triangles I . and II . of the moment area triangle and reducing their areas by means of the base $\frac{l}{2}$ to the lines I . and II . in the Mdy force polygon, thus giving another tangent at C . The tangents

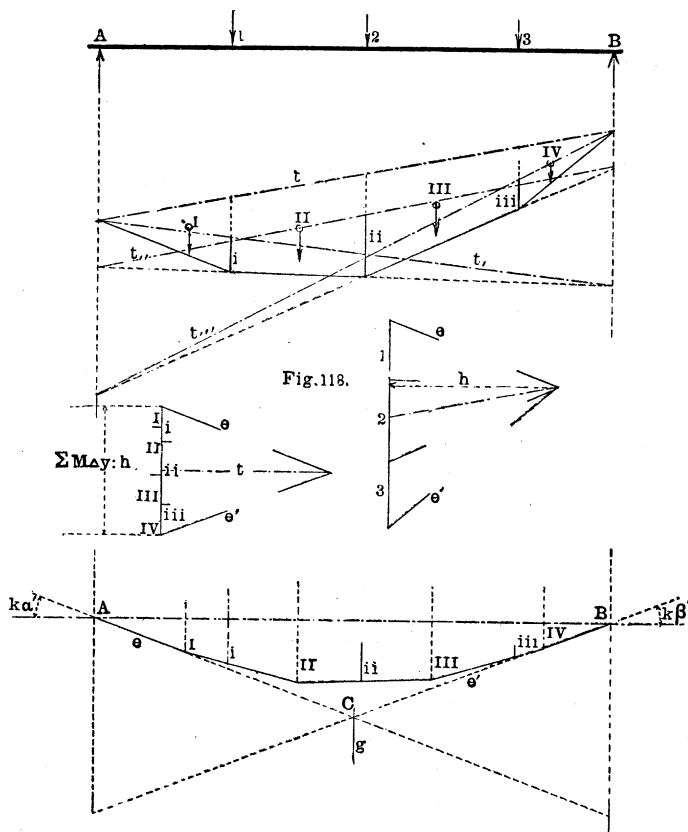
$$k \tan \alpha, k \tan \beta$$

are multiples of α and β in which

$$k = \frac{EI}{h \cdot \frac{l}{2} \cdot \frac{l}{3}}.$$

(b) An arbitrary number of weights upon the span (fig. 118).

Having formed the first force and cord polygons giving the complete moment area : h .



We can consider the first force and cord polygons as composed of separate force and cord polygons to each weight respectively, for instance, the force polygon of P_1 is composed of P_1 on the line of weights, ray e and $O(1, 2)$. Complete its separate cord polygon, as shown in figure, by extending the ray corresponding to $O(1, 2)$ to the vertical through B and drawing in its closing line t_1 . Complete in this manner the separate cord polygon of each weight, then as (38) any ordinate at any abscissa y of the combined cord

polygon is equal to the sum of the ordinates of the separate cord polygons at that abscissa, the sum of the areas of these separate cord polygons is equal to the area of the combined cord polygon, and if, in order to form the second or *Mdy* force polygon, we reduce the area of each of the separate cord polygons to the base $\frac{l}{2}$ then the thick ordinate *i* represents the moment area : *h*

due to P_1 . The thick ordinate *ii* represents the moment area : *h* due to P_2 The sum of these ordinates forms the exact length of the line of weights in the *Mdy* force polygon.

If we know beforehand, as *e.g.*, in symmetrical loading, the centre of gravity of the combined moment area : *h* cord polygon, draw through it a vertical *g* and transfer the lines *e*, *e'* of the *Mdy* force polygon (having a pole distance $\frac{l}{3}$) and find the tangential envelope *ACB*, giving

$$k \tan \alpha \quad \text{and} \quad k \tan \beta$$

as in the last case.

If the line *g* is unknown we may divide the moment area cord polygon by verticals through its angular points, find the centre of gravity *I*, *II*, *III*. of each (17), reduce their areas to the base $\frac{l}{2}$ and lay these reductions upon the line of weights of the *Mdy* force polygon, and the relation on the line of weights

$$I + II + III + IV + \dots (N + 1) = i + ii + iii + \dots n$$

gives a means of testing the accuracy of the reduction. (The provisional elastic line is not shown.)

We can now complete the tangential envelope cord polygon whose extreme rays necessarily (14) give a point in *g*. Verticals through *i*, *ii*. give the tangent points of the tangential envelope. As formerly

$$k = \frac{EI}{h \cdot \frac{l}{2} \cdot \frac{l}{3}}$$

145. *Construction of Elastic Line Tangent Envelope of an Unloaded Span, with Moments over the Supports arising from Forces Exterior to the Span.*—Let M_A , M_B have opposite signs. We

cord polygon, first provisionally, shown on fig. 116, in strong broken lines (art. 143, iii.), then find the pole S , and from it the tangential envelope to the given supports, supposed horizontal in figure shown in strong full lines.

The reducing base of the moment areas being $\frac{l}{2}$,

$$AA' \text{ and } BB'$$

represent these areas in the Mdy force polygon, and the pole distance being $\frac{l}{3}$ which is at the same time the distance G_A and G_B from the verticals through the supports, the intercept on the vertical through A of the first two sides of the envelope cord polygon measures

$$\frac{M_A}{h} \cdot \frac{l}{2}$$

and as the sides e, b of the cord are parallel to the rays e, b of force polygon with the vertical through A in the one case, and $\frac{M_A}{h} \cdot \frac{l}{2}$ on the line of weights on the other form two similar triangles each having a height of $\frac{l}{3}$, wherefore they are likewise equal and their bases are equal. In the same manner the intercept on the vertical through B cut off by b and e' of the envelope cord polygon measures

$$\frac{M_B}{h} \cdot \frac{l}{2}$$

As t and b in the elastic line envelope cut off intercepts on the verticals through supports equal to AA' and BB' , (or with another base than $\frac{l}{3}$, proportional to them) it follows from elementary principles that all t and b cut in the line x , which goes through the point of no bending moment of the elastic line, *i.e.*, at its point of inflexion under moment over the points of support, but under no load between supports.

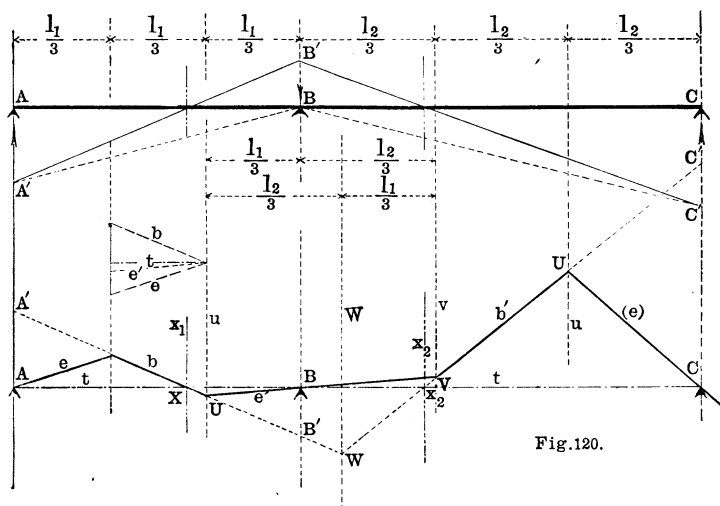
Corollary. The line through the point of inflexion of an

unloaded span, but having moments over the supports is the same as long as the ratio

$$\frac{M_A}{M_B}$$

between the moments over the two supports is the same, and is independent of their absolute value.

146. *Construction of Elastic Line Tangent Envelope for Two Adjacent and Unloaded Spans of a Continuous Beam* (fig. 120). —The continuity of the elastic line requires (143, v.) that $\tan \beta$ of span l_1 equal in value $\tan \beta$ of span l_2 over the support B but with opposite sign, that is, that the tangent line over a support be in the same straight line on both sides of that support.



From the last problem we have given

$$M_A \text{ and } M_B \text{ or } \frac{M_A}{M_B}$$

and the heights of the supports A and B , giving the elastic envelope in the opening l_1 , and consequently the tangent lines over supports A and B . Further, still looking upon the moment

areas as forces we have given us the vertical lines of action of the resultants, viz. :

$$u \text{ of } M_B \cdot \frac{l_1}{2} \text{ at a distance } \frac{l_1}{3}$$

to the left of B and

$$v \text{ of } M_B \cdot \frac{l_2}{2} \text{ at a distance } \frac{l_2}{3}$$

to the right of B .

Whatever be the actual value of M_B these two moment area forces

$$M_B \cdot \frac{l_1}{2} \text{ and } M_B \cdot \frac{l_2}{2}$$

are as l_1 to l_2 and

$$l_1 \cdot \frac{l_2}{3} = l_2 \cdot \frac{l_1}{3}$$

and the distance of their resultants from each other is

$$\frac{l_1}{3} + \frac{l_2}{3}$$

whence, dividing this distance inversely as the near force we obtain a point in the line of their resultant w .

Extending the line b of the elastic line envelope of the span l_1 till it cuts w in W and the line e' of the same span till it cuts v in V , we have now two points in the line b' of the elastic line envelope of the span l_2 , which extend till it cuts w of l_2 in U of l_2 , the line of action of the resultant

$$M_c \cdot \frac{l_2}{2}$$

giving a point in (e') of l_2 , thus giving the tangent line over support C .

This construction gives, by the same reasoning as in last problem, the vertical line x of inflexion of the elastic line in the span l_2 .

147. *Dependence of the Inflexion Points of a Series of Unloaded Spans upon the First Inflexion Point.*—The inflexion line x depends upon the ratio

$$\frac{M_A}{M_B}$$

the point X depending further on the relative height of the supports A and B . Let us take this last condition for granted, and for the sake of brevity let us say that the point X_1 depends upon the ratio

$$\frac{M_A}{M_B}.$$

From the above problem we perceive that the point X_2 depends upon the position of X_1 . From thence we perceive that the inflexion point of X_3 of a third unloaded span 1 depends upon X_2 whence there are a series of points in a series of spans depending upon the position of the first point X , and the position of X is such that

$$\frac{M_A}{M_B} = \frac{AX}{BX}.$$

Now in a continuous beam the moment over the first support A is zero, whence when A is the first support

$$\frac{M_A}{M_B} = \frac{0}{M_B} = \frac{0}{BX}$$

in which case therefore $AX = 0$, $BX = BA$ and X coincides with A , whence, in a continuous beam there are a series of points X , one in each span depending on the point A , the first support.

Again, the moment over the last support Y is likewise zero, whence we have, proceeding from that extremity of the beam, a series of points Y , of which the first coincides with the last point of support.

148. *A Geometrical Problem.*—Before proceeding further we must interpolate a geometrical problem. Changing the notation we quote the enunciation of the theorem (*Proj. Geom.*, art. xiii.). If the three summits of a triangle UVW (fig. 121) travel along three fixed lines u, v, w which concur in a point (in this case at infinity) whilst two of its sides UV, UW , pivot round two fixed points B and S , the third side WV pivots round a fixed point X in the same straight line with S and B .

Retaining U and V in any given position such as that in the

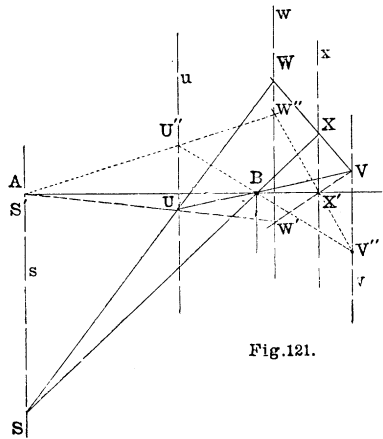


Fig. 121.

figure, we can again enunciate the same proposition thus. If the three summits of a triangle SWX travel along three straight lines s, w, x , which concur in a point (in this case at infinity) whilst two of its sides WX and SX pivot round two points B and V , the third side WS pivots round a point U in the line of B and V .

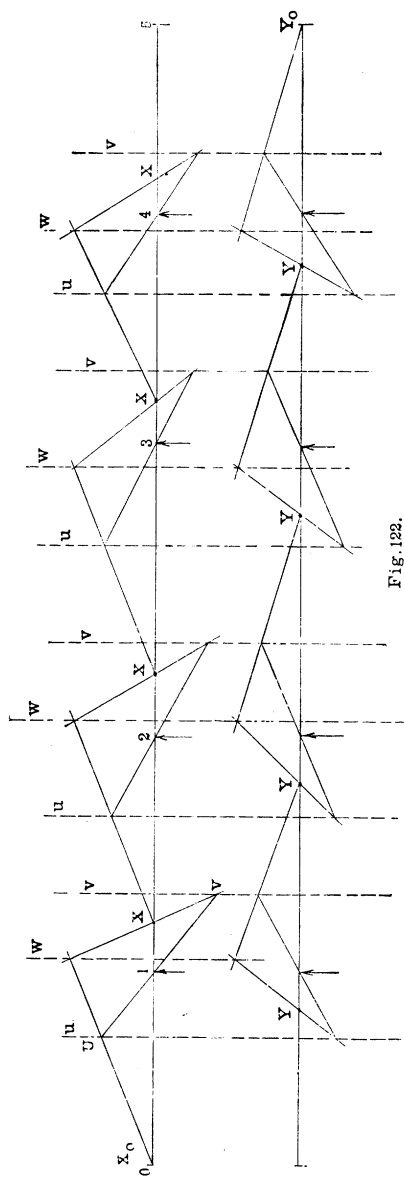
Let then WX of the triangle SWX pivot around V till its summits W and X have travelled along w and x into two new positions W' and X' and S has travelled along s to S' , then $S'W'$ by this proposition pass through U .

We can now enunciate the proposition again thus :

If the three summits of a triangle UVW' travel along three fixed lines u, v, w , which concur in a point, and if two of its sides UV and UW' pivot around two fixed points B and S' , the third side $W'V$ pivots around a fixed point X' in the same straight line with S' and B . The finely pointed triangle $U''V''W''$ is an arbitrary position of the triangle UVW' regarded as in this way varying.

From this chain of propositions we evidently deduce that the line x parallel to u, v, w (that is meeting them at infinity) depends only on the position of the point B and of the remaining lines s, u, v, w .

149. *Construction of the Inflexion Points of a Given Continuous Beam.*—We can now construct those two series of



points X and Y in a continuous beam referred to in (148), of which the first X coincides with the first point of support, and the first Y with the last point of support. Let 0, 1, 2, 3, 4, 5, (fig. 122) be the supports of a continuous beam of which the first X coincides with 0, and in which the verticals u, v, w have the distances assigned to them in fig. 120.

Draw a line otherwise arbitrary through the point 0, cutting u and w in U and W . Draw the line $U1$ cutting v in V , then WV will cut $\overline{01}$ in X , a point in the vertical X of the second span. Taking then this newly found X in place of 0, repeat the same construction for X in the third span, and so forth till we arrive at the last span of the beam.

Beginning again at the last support D , for the first Y carry out the same construction for all the Y series.

Corollary.—The points in the X series lie within the first third of the spans. The points in the Y series lie within the last third of the spans.

These two series of points are the points of inflexion of the elastic line with unloaded spans, and are called the *fundamental points* of the beam.

150. *Let there be given a Weightless Beam, having One Span only loaded, and with a Concentrated Weight P , to find the Bending Moments over the Supports, and the Elastic Envelope giving the Tangents over the Supports (fig. 123).*—Let AB be the loaded span in question. It is the span 2, 3 of fig. 122 drawn to a scale $\frac{4}{3}$, and x, y , the verticals through the fundamental points.

Let P be the weight and $AHB =$ the moment area $\div h$, then taking $\frac{l}{2}$ as formerly for reducing base

$$PH = i = \text{moment area} \div h \cdot \frac{l}{2},$$

thus forming the line of weights for the Mdy force polygon, for which choose as formerly $\frac{l}{3}$ for pole distance, and (S) as pole,

then were AB a free girder $A''GB''$ would be its elastic line envelope, and $A''C''$ would be

$$\Sigma_0^1 dy \Sigma_0^1 M dy \div h \cdot \frac{l}{2} \cdot \frac{l}{3} = \triangle AHB \cdot \gamma \div h \cdot \frac{l}{2} \cdot \frac{l}{3}$$

taking A for origin, and similarly $B''D''$

$$\triangle AHB \cdot \gamma' \div h \cdot \frac{l}{2} \cdot \frac{l}{3}$$

taking B for origin, for $A''G$ and $B''G$ are the extreme rays e and e' of the elastic line envelope regarded as a cord polygon.

Lay off on fig. 123(a) $AC = A''C''$, and $BD = B''D''$. Join AD , BC , and through the points X and Y where these two lines cut the verticals x and y , draw a line cutting the verticals through A and B in A' and B' , AA' and BB' measure the bending moments $\div h$ over the points of support.

For, supposing P to be removed, but the span transmitting the same moments but with opposite sign, owing to pressures right and left of it, then BB' is the bending moment $\div h$ transmitted from $A'C'$, which again has been transmitted from the left, all to the right being conceived unloaded. In the same manner AA' is the bending moment $\div h$ transmitted from $B'D$ which again has been transmitted from the right, all to the left being conceived unloaded.

But, if we suppose, as formerly, the triangles $AA'B$, $BB'A$, ACB , ABD reduced to lines by the base $\frac{l}{2}$, then AA' , BB' , AC , BD represent these triangles ($\div \frac{l}{2}$) and at the same time have the same moments on verticals through A and B , as the original triangle AHB has, for

$$\frac{\text{triangle } AHB}{\frac{l}{2}} \times \text{lever arm } \gamma \div \text{pole distance } \frac{l}{3} = A''C'' = AC$$

or

$$i \cdot \gamma \div \frac{l}{3} = AC$$

and

$$\frac{\text{triangle } ACB}{\frac{l}{2}} \times \text{lever arm } \frac{l}{3} \div \text{pole distance } \frac{l}{3} = AC$$

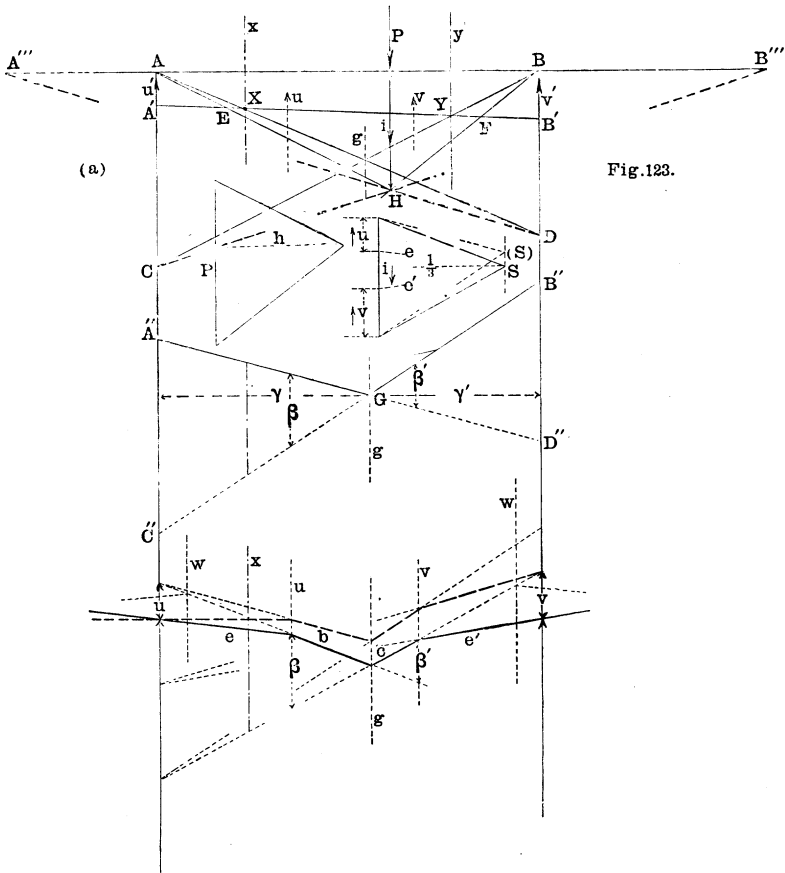


Fig. 123.

and

$$\text{triangle } ACB \div \frac{l}{2} = AC.$$

The points C and D may be a little more easily obtained by the following construction,¹ of which the student may easily find the demonstration. Lay off the span AB from P to A'' and from P to B'' (fig. 123a), and from these two points A'' and B'' draw lines through H , these lines will cut the verticals through A and B in C and D .

Whatever forces then, on either side of the span AB would

¹ Due to Prof. Lippich, Graz, *Theorie des kontinuierlichen Trägers konstanten Querschnittes*, Wien, 1871.

transmit to it these moments, would be equilibrated by the replacing of P ; P must therefore transmit equal moments but with opposite sign, and be equilibrated by these forces outside of the span.

We can now form the elastic line envelope of the span. Mark $u = AA'$ on the line of weights of the Mdy force polygon from its upper extremity downwards, $v = BB'$ from its lower extremity upwards (as our figure is arranged) this gives the rays e, e' , of the elastic line envelope considered as a cord polygon, from whence it may be constructed from an arbitrary pole (S) as shewn in a bold broken line, and then displaced (133, iii.) over the given support.

This problem has been carried out for one concentrated load P , but a consideration of problem (135, b) will shew that for this weight any number of concentrated weights or a uniformly distributed load may be substituted, the line g going through the centre of gravity of their moment area.

Corollary.—The moments over the supports of a loaded span are always negative.

151. *Remarks upon Choice of Constants.*—It is apparent that our choice of constants $\frac{l}{2}$ and $\frac{l}{3}$ has greatly facilitated the foregoing constructions, but in a continuous beam having spans of different lengths, the constants must be chosen proportional to one of those spans. Let l_s be such a standard span, and let it transmit, after the manner of (146) to any span l , (fig. 123)

$$(a) \ AA' = \frac{M_A}{h} = (\text{moment over support } A) : h \text{ and if } l = l_s$$

there follows that

$$(b) \ AA' \text{ is likewise} = \frac{M_A}{h} \cdot \frac{l_s}{2} : \frac{l_s}{2} \\ = (\text{moment area} : h) : \text{base } \frac{l_s}{2}$$

$$(c) \ AA' \text{ is likewise} = \frac{M_A}{h} \cdot \frac{l_s}{2} \cdot \frac{l_s}{3} : \frac{l_s}{2} \cdot \frac{l_s}{3} \\ = (\text{moment of moment area} : h) : \left(\text{base } \frac{l_s}{2} \times \text{pole distance } \frac{l_s}{3} \right).$$

But let l be greater or less than l_s , and, to fix the ideas, let it be the adjacent span, then

(a) AA' still represents $\frac{M_A}{h} = (\text{moment over support } A) : h$
for

$$\frac{M_A}{h} \cdot \frac{l_s}{2} \text{ is the (moment area) : } h$$

to the span l_s on the one side of the support, and

$$\frac{M_A}{h} \cdot \frac{l}{2} \text{ is the similar (moment area) : } h$$

to the span l_1 on the other side. But

(b) $\frac{M_A}{h} \cdot \frac{l}{2} : \frac{l_s}{2}$ is now the representative line of the

$$\text{moment area} = AA' \cdot \frac{l}{2} : \frac{l_s}{2}.$$

(c) $\frac{M_A}{h} \cdot \frac{l}{2} \cdot \frac{l}{3} : \frac{l_s}{2} \cdot \frac{l_s}{3}$ is now the representative line of the

$$\text{moment of moment area} = AA' \cdot \frac{l^2}{l_s^2}.$$

In this case $f = PH (= i \text{ of fig. 123a})$ no longer represents the moment area : $h = AHB$, but it is represented by

$$f \cdot \frac{l}{2} : \frac{l_s}{2} = f \cdot \frac{l}{l_s}.$$

whence, for the intercept $A''C''$ representative of the (moment of moment area AHB) instead of having (fig. 123)

$$\frac{l}{3} : i :: \gamma : A''C'' \quad \text{or} \quad A''C'' = \frac{3\gamma i}{l} \quad \text{or} \quad \frac{3\gamma f}{l} \dots (d),$$

we must have

$$\frac{l}{3} : f \frac{l}{l_s} :: \gamma : A_s''C_s'' \quad \text{or} \quad A_s''C_s'' = 3f \frac{\gamma l}{l_s^2}.$$

This intercept $A_s''C_s''$ is capable of a simple graphic construction. From (d) we have, not writing accents

$$3\gamma f = AC \cdot l \dots (\text{fig. 124}),$$

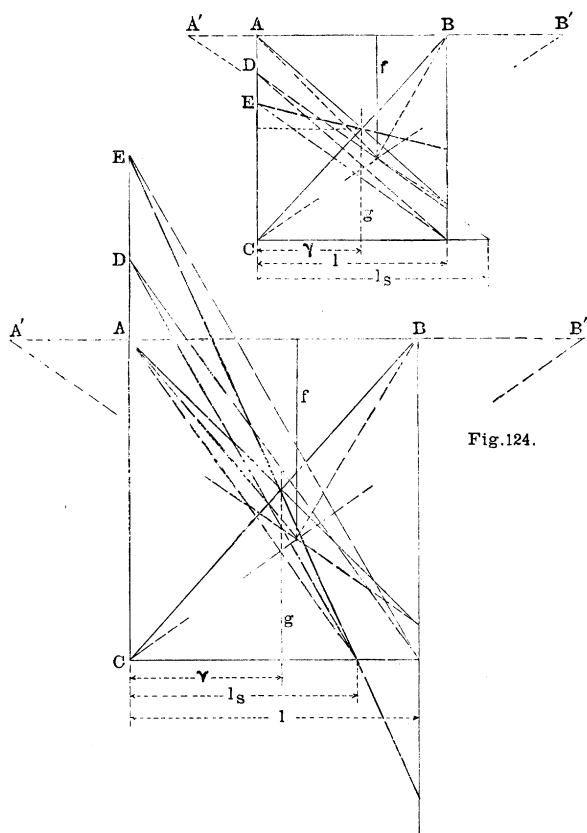


Fig. 124.

or, the right-angled triangle whose sides are

$3f$ and γ = a right-angled triangle whose sides are AC and l

$$3f\gamma \cdot \frac{l}{l_s} = AC \cdot \frac{l}{l_s} \cdot l = DC \cdot l.$$

This amounts to changing the base of $AC \cdot l$ to $DC \cdot l_s$, for, multiply the expression by $\frac{l_s}{l}$ and we have

$$AC \cdot l = DC \cdot l_s.$$

Again

$$3f\gamma \frac{l}{l_s} = DC \cdot \frac{l}{l_s} = EC : \dots (e)$$

This amounts to changing the base a second time, for, multiplying by l_s , we have

$$DC \cdot l = EC \cdot l_s$$

whence

$$EC = A_s C_s$$

the intercept required.

The student is by this time familiar with reducing one base to another, and the operations above are therefore simply represented in fig. 124 without more explanation. The examples given in this figure are the first and third spans of fig. 126.

152. *Moments over all the Supports of a Weightless Beam having only One Span Loaded, and its Elastic Line.*—Fig. 125 represents the continuous beam of fig. 122, and to the same scale. The moments over the supports of the loaded span are constructed after (141) fig. 123, and the moments over the other supports are constructed after problems 136 and 137, by carrying the oblique lines of the moments over the supports 2 and 3 through the points Y to the right of the loaded span, and through the points X to the left of it.

The remainder of the elastic line on either side of the loaded span (fig. 125a) can be constructed from problem of art. 146. The figure requires no further elucidation.

We can now transmit from any loaded span to any other span by means of the foregoing problems, and, taking any span, transmitting to it the moments over supports of all spans to right and to left, and summing them, we obtain the moments over its points of support.

153. *Example of a Continuous Beam with One Concentrated Load in each Span, Moments over Points of Support, and Elastic Line.*—We give an example in fig. 126 or Plate IV. of the employment of the foregoing principles and construction upon a beam over five openings symmetrically arranged round the central opening. One concentrated load has been placed arbitrarily upon each span, their force polygons have not been given. Fig. 126a gives the construction of the lines x and y . Figs. $b', b'', \dots b^v$ gives the load upon the $l_1, l_2 \dots l_5$ spans, and the distribution of the

bending moments over the supports. Fig. 126c gives the summation of these bending moments over the supports. Fig. 126d is the elastic line.

Span $l_2 = l_4$ has been chosen as the l_s or standard span, whence after (150) the intercepts u' and v' cut off by the interior sides of the elastic line in those spans regarded as a cord polygon is equal to the negative bending moment, and there being no other supports having bending moments there is data sufficient, by means of the lines u, v, w , to complete the elastic line. But had the beam not been symmetrical we should then have to reduce the intercepts of the load elastic line, for instance as in span 1, over support O to

$$3f_1 \cdot \frac{l_1\gamma}{l_s^2}$$

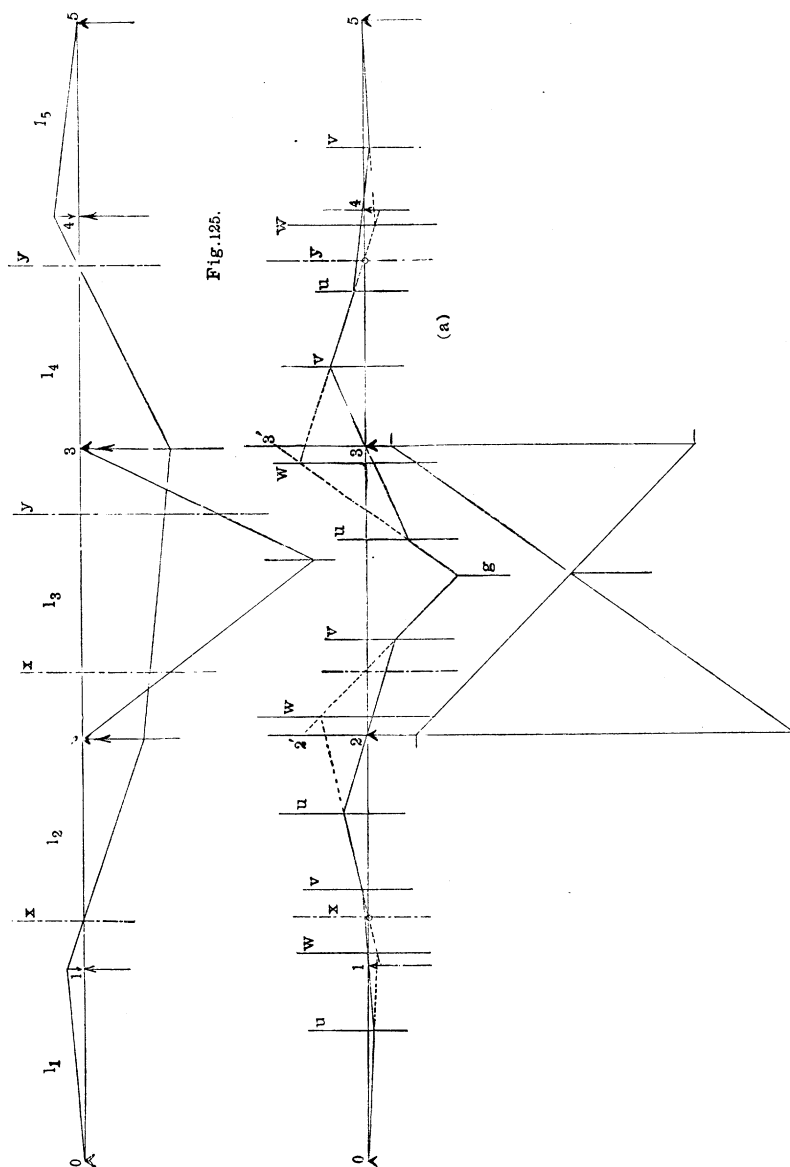
(this is worked out on fig. 124a), giving the point S_0 fig. 126d, fixing the point X_1 . The point S_1 again fixes the point X_2 , and the intercepts of the load elastic line of span l_3 must be reduced to

$$3f_3 \cdot \frac{l_3\gamma_3}{l_s^2}$$

giving the intercept X_2S_2 in the line x . The point S_0 being fixed, X_2 in the chain of points becomes fixed, whence lay off in fig. 126d X_2S_2 equal to X_2S_2 in fig. 126b'''. Were l_2 the only standard span, this chain of points must be carried on to the end of the beam, arriving at last at the point S_n^1 in our fig. S_5^1 . Such a variety in the length of spans would be a rare occurrence in practice, but it is, as is seen, easily solvable.

In fact by taking, in all cases, the second span of the bridge as standard span, we obtain the point S_1 from whence with any values of u and v we attain the point X_1 , then measuring the intercepts X_1S_1 , then $X_2S_2 \dots$ we proceed to the end, latterly obtaining $X_{n-1}S_{n-1}$, in our figure X_4S_4 . We have then two points n and S_{n-1} (5 and S_4) in the concluding line of the elastic envelope, giving two points in X_{n-1} and V_n in the next line.

This method of finding the points by any values of U and V has been employed in fig. 127d, where DS of fig. 127d, is necessarily parallel to DS of fig. 127b.



154. *Extension of Problem to a Number of Concentrated Loads in any Spans.*—We have taken a single concentrated weight P over each span as being sufficient for our present purpose, but had we taken a series of concentrated loads, the f of fig. 126 becomes the (i. + ii. + iii.) of fig. 118. The verticals g (fig. 126) through the centres of gravity of the triangles $OH1$. $1H2$ are the verticals through the positive moment area, as that of fig. 123. These concentrated loads might be the weights of a series of locomotives, and the moments over the supports might be solved in relation to them with little more trouble than for a uniformly distributed load.

155. *Simplifications introduced from Uniform Loading.*—In uniform loading the moment area is a parabola, which may be supposed to have been formed from a force polygon with some pole distance h . Let this pole distance be some multiple of the depth h' of the girder, or

$$h = nh'.$$

If p is the load per unit of length, then the positive bending moment at the centre of the span is

$$p \frac{l}{2} \cdot \frac{l}{2} - p \int_0^{\frac{l}{2}} d \cdot dy = p \frac{l^2}{4} - p \cdot \frac{l}{2} \cdot \frac{l}{4} = p \frac{l^2}{8}.$$

This bending moment : nh is represented by the ordinate of the parabola at the centre of the span, *i.e.*,

$$f = \frac{p \cdot \frac{l^2}{8}}{nh'}.$$

The moment area : nh is

$$\frac{2}{3} l \cdot p \cdot \frac{l^2}{8} \cdot \frac{1}{nh'} = \frac{2}{3} \cdot l \cdot f$$

and let, as formerly $\frac{l}{2}$ be the reducing base of the area, *i.e.*, let area be represented by

$$\frac{2}{3} \cdot \frac{lf}{\frac{l}{2}} = \frac{4}{3} f.$$

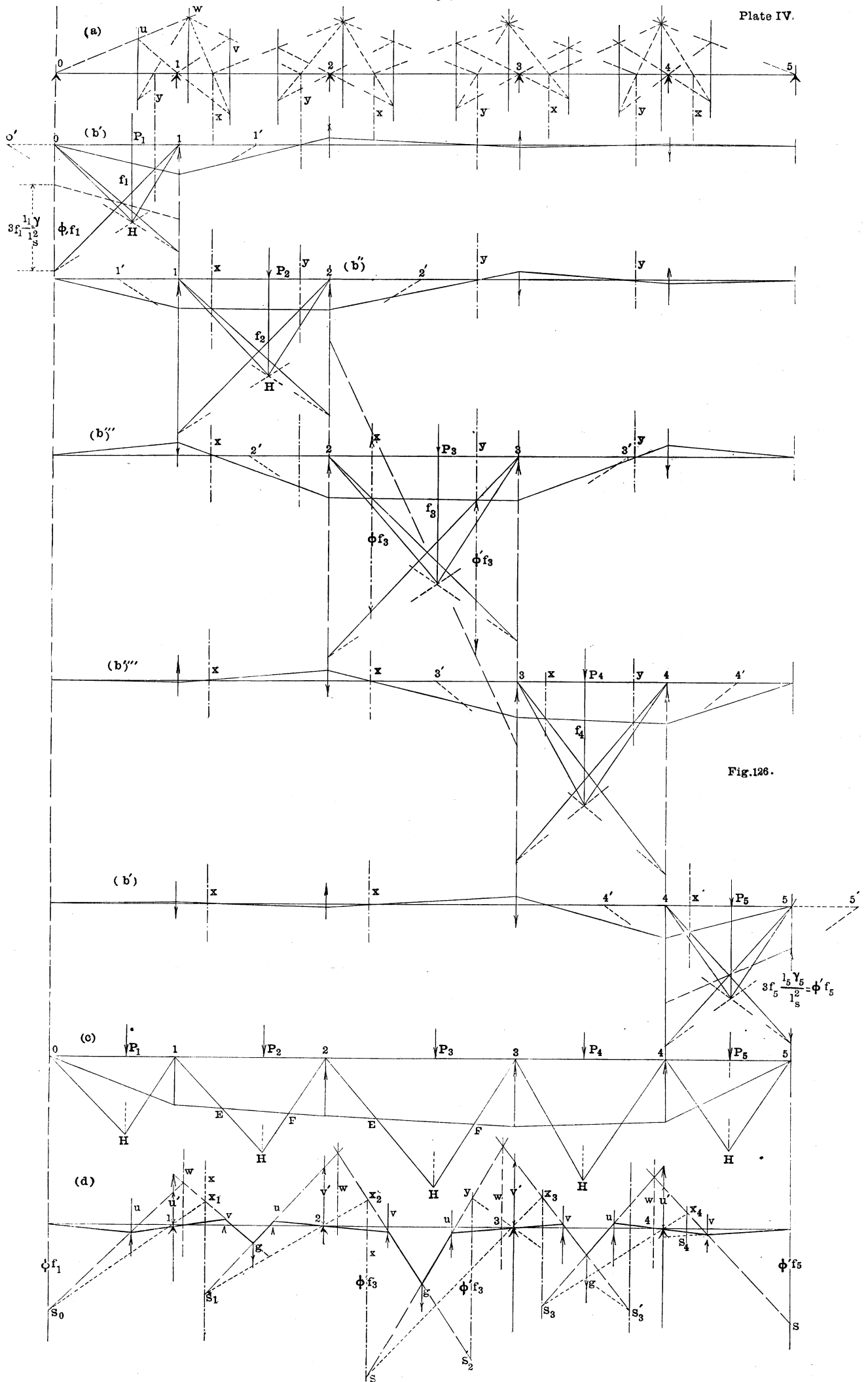


Fig. 126.

γ is in this case $= \frac{l}{2}$, and let, as formerly, $\frac{l}{3}$ be the pole distance for the elastic envelope

$$\begin{aligned} & \text{representative line of moment area} \times \gamma \div \text{pole distance } \frac{l}{3} \\ &= \frac{4}{3}f \cdot \gamma \div \frac{l}{3} = \frac{4}{3}f \cdot \frac{l}{2} \cdot \frac{1}{l} = 2f. \end{aligned}$$

This then is the intercept for the standard span (see fig. 127c), central or standard span.

For spans whose intercept requires to be reduced

$$3f \cdot \frac{l}{l_s} \gamma \text{ becomes } 2f \cdot \frac{l^2}{l_s^2}$$

for

$$\frac{\text{moment area}}{\frac{l_s}{2}} \times \text{lever arm } \gamma \left(= \frac{l}{2} \right) \div \text{pole distance } \frac{l_s}{3}$$

$$\frac{\frac{2}{3}fl}{\frac{l_s}{2}} \cdot \frac{l}{2} \div \frac{l_s}{3} = 2f \frac{l^2}{l_s^2} = \phi f.$$

We may either construct or calculate these intercepts, then lay them off on the verticals through the corresponding supports crossing in a point in the vertical g (fig. 127c) and proceed as before.

156. *Example of Continuous Beam with Uniform Loading.*

—Take for illustration a continuous beam over three openings the dimension and loading being as under, two of the openings being supposed loaded, and the third with structural load only.

$$l_1 = 120 \text{ feet, } p = 2 \text{ tons, } p \frac{l_1^2}{8} = 3600, nh' = 40 \text{ gives } f_1 = 90.00.$$

$$l_2 = 150 \text{ feet, } p = 2 \text{ tons, } p \frac{l_2^2}{8} = 5625, nh' = 40 \quad ,, \quad f_2 = 140.62.$$

$$l_3 = 120 \text{ feet, } p = 1 \text{ ton, } p \frac{l_3^2}{8} = 1800, nh' = 40 \quad ,, \quad f_3 = 45.00.$$

Intercept for $l_1 = 2f_1 \cdot \frac{l_1^2}{l_2^2} = 115.20 = \phi f_1$ for brevity.

„ „ $l_2 = 2f_2 = 281.25 = 2f_2$ „ „

„ „ $l_3 = 2f_3 \cdot \frac{l_3^2}{l_2^2} = 57.60 = \phi f_3$ „ „

In the case of the standard span l_2 we know that BB' , CC' (fig. 127*b*) measure directly

$$\frac{M_B}{nh'} \text{ and } \frac{M_C}{nh'}$$

and can be transferred to BB' , CC' , of fig. 127*a*. The elastic envelope (fig. 127*d*) can be drawn in after art. 153, and requires no further elucidation.

157. *Uniform Distribution of Pressure over Part of a Span.*—Let uniform pressure (fig. 128) extend over the span $AB = l$, for a length $= \beta l$, then $p\beta l$ is the total amount of pressure.

The reaction at B is

$$\frac{1}{2}\beta \cdot p\beta l = \frac{1}{2}\beta^2 pl.$$

The moment of the load at its extreme point C

$$\begin{aligned} &= \text{Reaction } B \times BC = \frac{1}{2}\beta^2 pl(1 - \beta)l \\ &= \frac{1}{2}\beta^2(1 - \beta)l^2 p. \end{aligned}$$

Lay this $\div nh$ off as an ordinate CDH of the moment area, BD is one of the extreme rays e' of the moment area considered as a cord polygon. The intercept of e' upon the vertical through A is

$$\frac{1}{2}\beta^2 \cdot l^2 p.$$

The central ordinate of the parabola is

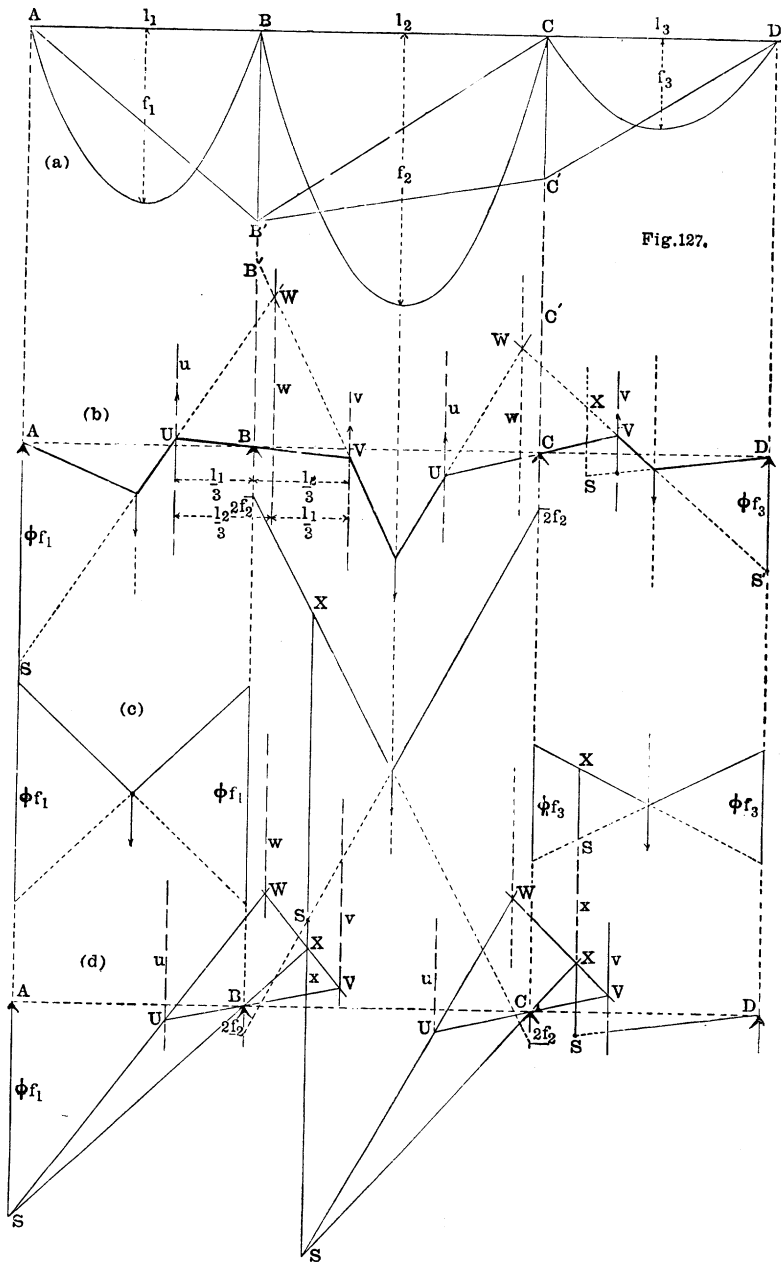
$$\frac{1}{8}\beta^2 \cdot l^2 p.$$

The contents of the parabolic portion A_1 of the moment area is

$$\frac{1}{12}\beta^3 \cdot l^3 p.$$

The contents of the triangular portion, ADB or A_2 , is

$$\frac{1}{4}\beta^2(1 - \beta)l^3 p.$$



The contents of the whole moment area is

$$\frac{1}{12}\beta^2(3 - 2\beta)l^3p.$$

The moments of these areas around A is:

For the parabolic area

$$\bar{M}'_A = \frac{1}{2}\beta l \cdot A_1 = \frac{1}{24}\beta^4 \cdot l^4p.$$

For the triangular area

$$\bar{M}''_A = \frac{1}{3}(1 + \beta)l \cdot A_2 = \frac{1}{12}\beta^2(1 - \beta^2)l^4p.$$

The total moment around A

$$= \bar{M}_A = \frac{1}{24}\beta^4(2 - \beta^2)l^4p.$$

The total moment around B

$$= \bar{M}_B = l(A_1 + A_2) - \bar{M}_A = \frac{1}{24}\beta^2(2 - \beta^2) \cdot l^4p.$$

For $\beta = 1$ we have as we otherwise know

$$\bar{M}_A = \bar{M}_B = \frac{1}{24}l^4p.$$

The intercepts, then, of the extreme sides of the elastic line upon the verticals through the points of support have for co-efficients, of

$$\frac{1}{24}pl^4 = 2f.$$

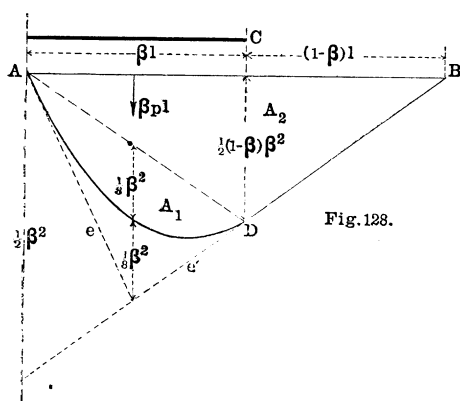
For the supports adjoining the load

$$\beta^2(2 - \beta^2).$$

„ „ „ furthest from the load

$$\beta^2(2 - \beta^2).$$

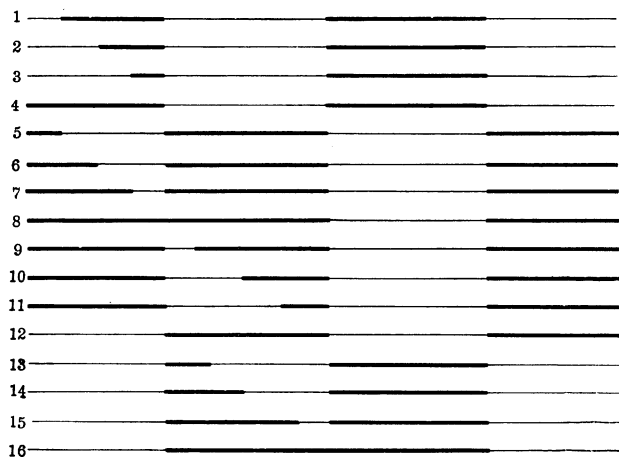
As these co-efficients will be constantly required in the consideration of any case, we give their values in the adjoining table for given values of β :



| Fraction β of Span. | Coefficients adjoining Load. | Coefficients distant from Load. |
|------------------------------|---------------------------------|------------------------------------|
| β . | $\beta(2 - \beta^2)$. | $\beta(2 - \beta)^2$. |
| 0.1 | 0.0199 | 0.0361 |
| 0.2 | 0.0784 | 0.1296 |
| 0.25 | 0.1211 | 0.1914 |
| 0.3 | 0.1719 | 0.2601 |
| 0.4 | 0.2944 | 0.4096 |
| 0.5 | 0.4375 | 0.5625 |
| 0.6 | 0.5904 | 0.7056 |
| 0.7 | 0.7399 | 0.8281 |
| 0.75 | 0.8086 | 0.8789 |
| 0.8 | 0.8704 | 0.9216 |
| 0.9 | 0.9639 | 0.9801 |
| 1.0 | 1.0000 | 1.0000 |

158. *Scheme of Loadings to be considered in the Determination of the Maximum Positive and Negative Moments.*—We are now in a position to find the negative and positive moments for given varieties of loading, and from thence, by inspection, to find the maximum negative and positive moments over the beam. For instance, in a beam of four spans, the sixteen different states of loading shown in the following scheme (fig. 129), where a thick line denotes loading, would require to be more or less completely constructed, so far as to get the envelope of the moments arising from the combined loadings.

Fig. 129.



159. *Ratio of Ordinates of Constructed Elastic Line to those of True Elastic Line.*—In fig. 127 we have obtained an elastic line whose equation is

$$\frac{1}{nh'} \cdot \frac{1}{\bar{l}_s} \cdot \frac{1}{\bar{3}} \int_0^y dy \int_0^y M dy = \frac{6}{nh' \bar{l}_s^2} \int_0^y dy \int_0^y M dy,$$

and substituting the symbol i for the intercept

$$\frac{\Sigma m z^2}{abc}$$

we have $I = abci$, and as the true elastic line is

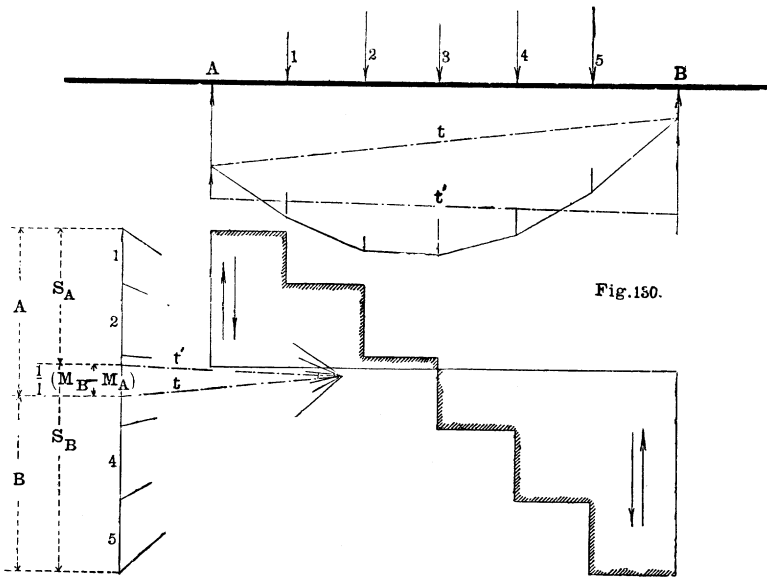
$$\frac{1}{EI} \int_0^y dy \int_0^y M dy,$$

we have obtained an elastic line whose ordinates are to those of the true elastic line as

$$6EI : nh' \bar{l}_s^2,$$

or as

$$6Eabci : nh' \bar{l}_s^2.$$



160. *Shearing Force* (fig. 130).—The equation to shearing force over points of support in any span is

$$S_A = \frac{1}{l}(-M_B + M_A + \Sigma_0^l P(l-a)).$$

The part

$$\frac{1}{l} \Sigma_0^l P(l-a)$$

of this expression is the reaction of a free girder at A (36) a being the abscissa of a weight regarding A as origin. The notation of art. 36 being such that $x = (l-a)$, and can be constructed according to (21) or (36). The part

$$\frac{1}{l}(M_B - M_A)$$

is the part intercepted between the lines t and t' (fig. 130), for

$$l : \left(\frac{M_B - M_A}{h} \text{ from cord polygon} \right) :: h : \frac{M_B - M_A}{l}$$

whence the construction of $S_A, S_A - 1 \dots S_B$ in the figure, remembering that $\frac{M_A}{l}$ adds to the shearing S_A and $\frac{M_B}{l}$ deducts from S_A . In fact t' in force polygon decides the shearings S_A and S_B as t in the same decides the reactions A and B .

The shearing in any span can thus be determined.

161. *Most Unfavourable Pressures upon a Continuous Beam.*—For the sake of simplicity, let us consider the beam as weightless and take into account varied loading only, and divide the investigation into two parts.

I. *A Chosen Span Loaded.*—We have already determined the moments and shearing forces arising from a concentrated pressure within the span, and as a continuous load may be regarded as a series of infinitely small loads placed at infinitely near cross sections, and as in the girder free upon its supports we can conclude at any cross section τ the shearing is a negative maximum $^P\downarrow_s$, if the left part a of the span is loaded and a positive maximum $^s\downarrow^P$ if the right part of the span ($l - a$) is continuously loaded.

i. *Condition of Maximum Moment for Cross Sections between Fundamental Points X and Y.*—Consider fig. 123, p. 217. As A''' always necessarily lies beyond A , and B''' always lies beyond B outside of the span AD and BC fixing the points C and D , X and Y always lie within $A'''H$ and $B'''H$, whence the points E and F always lie beyond the points X and Y , and as EFH is the excess of the positive over the negative moments, so it follows that for every cross section between X and Y there is a positive moment for any given position of a concentrated load, whence follows: in every cross section between the fundamental points X and Y the moment is a positive maximum, if the whole span is continuously loaded.

ii. *Condition of Maximum Moment for all Cross Sections between Supports and Fundamental Points of Span, i.e. for all cross sections σ in AX and BY .*—Consider in fig. 123, any cross section τ between Y and B , we see that if a pressure P lies infinitely near A , the inflexion point F coincides with Y . Let

P move forward from A toward B , F also moves toward B . There is then a position π for P for which F coincides with τ and for all pressures between A and π the moment in τ is negative, on the other hand pressures between π and B generate in τ a negative moment. We may take in the same manner a cross section τ' between A and X , hence the following theorem:

In any cross section τ between one of the fundamental points X or Y and the nearest support A or B the moment is a negative maximum when the pressure extends from A or B over the beam to a point π , for which, as point of action of a concentrated pressure the cross section under consideration is an inflexion point and a positive maximum when the pressure covers the beam from A or B to that cross section τ .

II. *A Chosen Span Unloaded.*

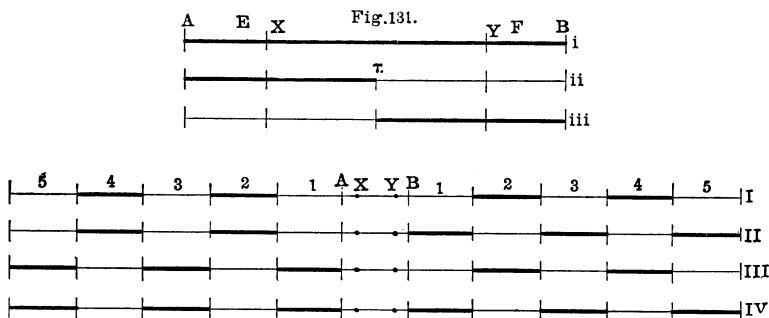
i. *Maximum Shearing Force.*—In an unloaded span, the shearing force is necessarily constant, and in order that it may be a maximum, the oblique line $A'B'$ (fig. 119) which unites the extreme points of the moment ordinates over the supports, forms with the beam the greatest angle possible, for, AB corresponds to t in the force polygon and $A'B'$ to one of its rays t' (fig. 130) and between them intercepting the shearing force on the line of weights.

This will happen when the moments over the supports have the greatest possible values but opposite in sign.

The shearing force at A is $s \uparrow \downarrow^P$ positive when the moment at A is negative, and inversely.

ii. *The Moments over a Support become a Maximum when the Moments over that Support arising from Pressure in Other Spans have the Same Sign.*—Now the reduced moments in two successive supports have opposite signs, so the first condition for a maximum is, when loaded spans alternate with unloaded, and the second condition is, when the loaded spans are completely decked over with their loads.

From the span AB (fig. 131*b*) mark to the right and to the left, separately, the remaining spans with the numbers 1, 2, 3 . . . so we see that over A , for example, there will be a negative maximum moment, if the odd spans to the left and the even



spans to the right are fully loaded (fig. 131, III.). The same pressures give a positive maximum moment over B , whence

The shearing force within AB will, with pressures exterior to it, be a $s \uparrow \downarrow^P$ positive maximum when the odd spans to the left and the even spans to the right are fully loaded, inversely a $P \downarrow \uparrow_s$ negative maximum.

ii. *Negative Maximum Moment from Support A to Fundamental Point Y .*—In regard to the moments, consider, especially, a negative maximum. Now in an unpressed opening the moment curve is a straight line, so the maximum coincides with certain maxima of the moments over supports.

Let, first, the moment over A be a negative maximum arising from pressures to the left of the span, so for this moment must all odd spans to the left be fully pressed (fig. 6, 131, III.). Within AB the curve of moments will be a straight, s cutting AB in Y and the moments are negative from A to Y . Now, bring forward the influence of pressures from the right, then X is the inflexion point of the right-hand pressures and the previously existing moment in X is not altered, for the influence of exterior (to the span AB) right-hand pressures upon X as of exterior left-hand pressures upon Y is zero. Let now the influence of pressures be brought from the even spans to the right, this influence generates in B a positive and consequently in A a negative moment, the straight s now turns in a negative direction, *i.e.* contrary to the hands of a watch, and the moments from A to X increase. Hence, they reach their maximum when s has reached its outermost position s' , and this is the case when all the odd spans to the left and the even spans to the right are fully loaded.

This condition is the same as for the maximum shearing force, for in A we have the greatest negative, and in B the greatest positive moments (i.), so that in A we have likewise the greatest negative $P\downarrow s$ shearing effort, and on B the greatest positive $s\uparrow P$ shearing effort, and inversely (i.).

Let, now, the influence of pressures be brought from the odd spans to the right, these generate in B , a negative moment, and s turns in the positive direction, *i.e.* in the direction of the hands of a watch; and the moments between X and Y increase negatively, and there is a negative maximum between X and Y when all odd spans to the right, and all even spans to the left, are fully loaded.

162. Recapitulation of Most Unfavourable Conditions of Loading, fig. 131.

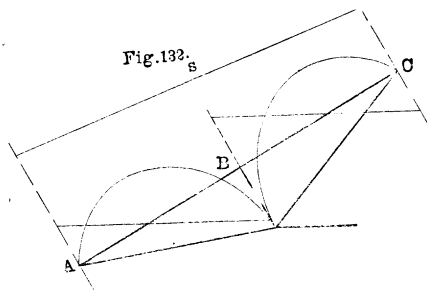
- Scheme i. Positive max^m. bending moment between X and Y .
 „ ii. Negative shearing maximum $P\downarrow s$ at τ .
 „ „ Positive max^m. bending moment in E .
 „ „ Negative max^m. bending moment in F .
 „ iii. Positive shearing maximum $s\uparrow P$ at τ .
 „ „ Positive maximum bending moment in F .
 „ „ Negative maximum bending moment in E .
 „ I. Positive max^m. bending moment between X and Y .
 „ II. Negative max^m. shearing force $P\downarrow s$ from A to B .
 „ „ Negative max^m. bending moment between Y and B .
 „ „ Positive max^m. bending moment between A and X .
 „ III. Positive max^m. shearing force between A and B .
 „ „ Positive max^m. bending moment between Y and B .
 „ „ Negative max^m. bending moment between A and X .
 „ IV. Negative max^m. bending moment between X and Y .

Combining Schemes i. and I., ii. and II., iii. and III., i. and IV., we obtain total maxima.

163. *Maximum Value of Reaction over a Point of Support as A.*—The reaction at any point of support as *A* is equal to the difference between the shearing force immediately to the right and left of it. Hence it is a negative or positive maximum according as the moment over the point of support is a positive or negative maximum.

164. *Structural Weight of Beam.*—It is only necessary to add the shearing forces and bending moments arising from the structural weight to these schemes, which can be made a separate scheme.

165. *Influence of the Height of the Points of Support.*—In our method no hypothesis has been made in regard to the heights of the points of support above a level line, and the constructions are therefore true when these are not all on the same level.



166. *Case of Continuous Beam with Varying Longitudinal Force.*—When the French roof of fig. 51 has only main trussing, one side of it is a continuous beam over three supports with an oblique force increasing from the apex of the roof to the support at the wall, and having negative bending moments at the central support (fig. 132). The ordinates between the roof *ABC* and the line *s* are the values of compressive force at their respective points. The ordinates to the parabolas are the values of bending moments, and the ordinates to the oblique lines across the spans *AB* and *BC* are the shearing forces.

In designing the beam it is necessary to find the neutral axis for a number of points after the method of 115, fig. 89.

167. *Loaded Beam Inbuilt at A and B (fig. 123) in given directions.*—This case is equivalent to having the points *A* and *B* given (not necessarily on the same level), the directions *c* and *c'* in which the inbuilding has been made, given, the line *g* given, and

$\Sigma m \cdot dy : h$ (i.e. *i* of fig. 123 or (*i* + *ii* + *iii*) of fig. 118) given, to find the moments *U'* and *V'* over the points of support : *h* so that the lines *b* and *c* may cut in a point in *g*.

This can be done by making the intercepts β, β' , upon *u* and *v* in fig. 123*b* equal to β and β' in fig. 123*c*, and through the end points of β and β' drawing in *b* and *c*, *b* and *c* will cut off on the verticals through supports the moments required. After what has preceded, this requires no formal demonstration.

168. *Equation of Elastic Line to a Beam of Variable Cross Section.*—The moment of inertia *I* of a beam varies with its cross section, and the form of equation (2), art. 143, must be modified by putting *I* within the sign of summation, thus

$$\frac{dz}{dy} \mp \text{constant} = \pm \frac{1}{E} \int \frac{M}{I} dy,$$

or in finite differences

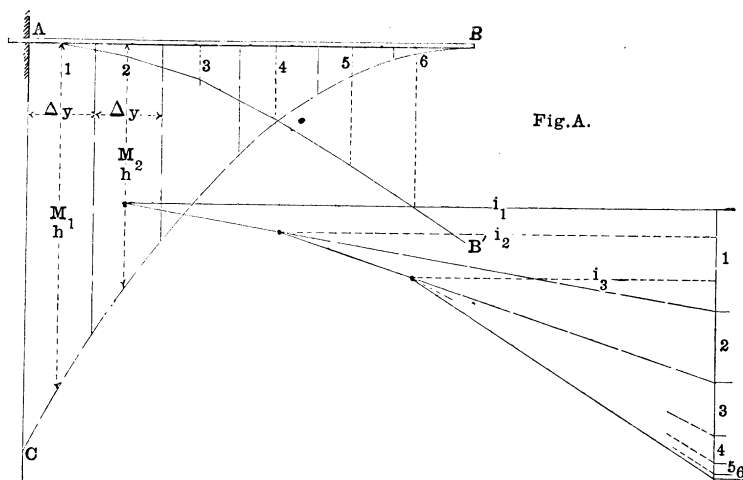
$$\frac{\Delta z}{\Delta y} \mp \text{constant} = \pm \frac{1}{E} \Sigma \frac{M}{I} \Delta y,$$

whence for the *Mdy* force polygon, we can no longer use *EI* or a multiple of *EI* as pole distance.

169. *Elastic Line of a Loaded Beam, inbuilt at one end, free at the other, and having a Variable Cross Section.*—Let *AB* (fig. *A*) be such a beam inbuilt at *A*, say horizontally, and loaded uniformly

$$ABC = \text{moment area} : h ; \Delta y_1, \Delta y_2, \Delta y_3,$$

certain lengths of the beam for which the cross sections give *I*₁, *I*₂, *I*₃ for the corresponding moments of inertia, whose intercepts on the second moment cord polygons of the three cord polygons of the three cross sections are *i*₁, *i*₂, *i*₃. Lay off upon the line of weights of the *Mdy* force polygon



$$\frac{M}{h} \Delta y_1, \quad \frac{M}{h} \Delta y_2, \quad \frac{M}{h} \Delta y_3, \quad \frac{M}{h} \Delta y_4 \dots$$

then we employ i_1, i_2, i_3 successively as pole distance, as shown, in forming the elastic line AB' , whose ordinates from the horizontal are in the proportion of

$$\frac{EI}{hi}$$

to the true elastic line.

170. *Influence of Variable Cross Section upon the Stresses of a Continuous Beam.*—In designing a beam to sustain the maximum stresses determined under the supposition of uniform cross section, we necessarily introduce a variable cross section, and consequently a variable I , and this again increases the stresses in certain distributions of load and decreases them in others. This is owing to the displacement of the points of contrary flexure E and F . Let there, for example, be given a continuous beam of uniform cross section, and under some given load in one of the spans, let E and F be the points of contrary flexure, then let material be taken from the cross section over and near to one of the points of support, then the radius of curvature of the beam over the support is necessarily diminished, and the point of contrary flexure moves towards A . Let again material

be taken from the cross sections near the middle C of the beam, the radius of curvature for that place is diminished and the two points of contrary flexure move toward C . The point of maximum moment lies between the two points E and F and nearest to that one of the two which has been moved the most. The alterations in the moments are the greater, the greater the pressure.

In rare instances, however, is it considered necessary to amend the design.

171. *Determination of the v and u Lines in a Continuous Beam of Variable Cross Section (fig. B).*—These v and u lines in a beam of uniform cross section are (145) lines through the centre of gravity of the positive and negative moment areas

$$\Sigma_0^l M dy \text{ and } \Sigma_0^l M' dy$$

of an unloaded span and are simply $\frac{l}{3}$ distant from each support, and we had, from the property of the centre of gravity

$$\frac{l}{3} \Sigma M dy = \Sigma M y \cdot dy = \text{moment of moment area around support.}$$

In this case, however, the areas of an unloaded span are

$$\Sigma_0^l \frac{M}{I} dy \text{ and } \Sigma_0^l \frac{M'}{I} dy$$

and calling their respective distances from their corresponding supports η and η' we have

$$\eta \cdot \Sigma \frac{M}{I} dy = \Sigma \frac{M}{I} y \cdot dy = \text{moment of moment area around support.}$$

The values of η and η' are, on elementary grounds, independent of particular values of M_A and M_B over supports, whence, in order to obtain them, we may assume any values. Let then AB be an unloaded span, having brought upon it from the left a moment M_A over the support A , generating, let us say, a moment M_B of opposite sign over the support B .

Over the span are laid i_1, i_2, i_3 proportional to the I in the parts of the span over which they are laid. Let, as formerly, the moment area : h

$$= (M_A : h) \frac{l}{2} = AA'B,$$

be represented by a line to the base $\frac{l}{2}$, then we have

$$(M_A : h) \cdot \frac{l}{2} : \text{pole distance } \frac{l}{2} = M_A : h = AA'.$$

Divide $AA'B$ vertically into trapeziums at distances corresponding to the changes in i . The vertical lines, 1, 2, 3, along the mean breadth of the trapeziums represent the area of these trapeziums to the base $\frac{l}{2}$ so that $1 + 2 + 3 = AA'$. In the same manner $1' + 2' + 3' = BB'$. Lay off these lines 1, 2, 3, as a line of weights to an

$$\frac{M}{hi} \cdot \Delta y$$

force polygon (fig. *b*). From this line of weights lay off horizontally i_1, i_2, i_3 , draw the first ray e (wrongly marked c on figure) of the force polygon, cut off at a horizontal distance i_1 from the line of weights in the point $1''$. From this point $1''$ draw the second ray $\overline{1''(1, 2)}$ of the force polygon, which cut off at a horizontal distance i_2 from the line of weights in the point $2''$, from whence draw the next ray $\overline{2''(2, 3)}$. In this manner proceed with the other rays till we arrive at the highest ray b . This procedure is evidently similar to the formation of the force polygon in art. 169. The first part of the force polygon, of which e and b are the extreme rays, is now complete, being the $\frac{M}{I} dy$ force polygon whose line of weights is

$$\Sigma_0^i \frac{M}{hi} \Delta y.$$

The $\frac{M'}{I} dy$ force polygon whose line of weights is $\Sigma_0^i \frac{M'}{hi} \Delta y$ will be treated as the second part of a complete $\frac{M}{I} \Delta y$ force polygon, but it is drawn in as a separate figure (fig. *c*) in order to avoid confusion of lines. But in order that it may form part of fig. (*b*) the line of weights and the rays, e and b , are equal and parallel.

The rays now succeed each other from b downwards, the same process of formation being observed with the lines $3', 2', 1'$.

With these force polygons we must now form cord polygons in order to find the centres of gravity of two ideal laminae ABA' , ABB' divided into trapeziums whose densities are as their areas divided by their corresponding i . Whence drawing verticals $g_1, g_2 \dots$ through the centres of gravity of these laminae, with the corresponding rays of the force polygons form cord polygons, whose sides cut in their appropriate verticals $g_1, g_2 \dots$ (figs. (d) and (e)). The intersections of their extreme rays e and b , and b and e' give a point in v and u respectively.

172. Formation of an Elastic Line Envelope of an Unloaded Span.—Extend the line e' till it meets the vertical through B , then from this point to the point where e meets the vertical through A , draw the closing line t . Where t meets b is a point in the fundamental line y .

Our procedure gives us, for intercept on vertical through support of cord polygon. For example, let τ' represent representative line of trapezium 1, then we have

$$i_1 : \tau' :: y_1 : \frac{\tau'}{i_1} \cdot y_1$$

: intercept on a ,

so that the summation of the whole intercept is

$$\sum \frac{\tau}{i} \cdot y = \sum \frac{M}{h} \cdot y \cdot dy = \eta \cdot \sum \frac{M}{h} \cdot \frac{1}{i} dy.$$

Let us take i out of the symbol of summation.

e and b in the force polygon cut in the point (e, b) at a distance from the line of weights, which call i_a .

e' and b in the force polygon cut in the point (e', b) at a distance from the line of weights, which call i_b .

i_a and i_b form pole distances true for the extreme lines of the cord polygons. Whence

$$\sum \frac{M}{h} y dy = \frac{\eta}{i_a} \sum \frac{M}{h} dy,$$

and

$$\sum \frac{M'}{h} (l - y) dy = \frac{\eta'}{i_b} \sum \frac{M'}{h} dy.$$

Now it is easily seen that

$$\frac{M}{h} = \frac{M_A}{h} \cdot \frac{l-y}{l} \quad \text{and} \quad \frac{M'}{h} = \frac{M_B}{h} \cdot \frac{y}{l},$$

whence substituting

$$\begin{aligned} \Sigma \frac{M}{hi} \cdot y \cdot dy &= \Sigma \frac{M_A}{h} \cdot \frac{l-y}{l} \cdot \frac{y}{i} \cdot dy = \frac{\eta}{i_a} \Sigma \frac{M}{h} dy \\ \Sigma \frac{M'}{hi} (l-y) dy &= \Sigma \frac{M_B}{h} \cdot \frac{y}{l} \cdot \frac{l-y}{i} \cdot dy = \frac{\eta'}{i_b} \Sigma \frac{M'}{h} dy. \end{aligned}$$

Now evidently

$$\frac{M_A}{M_B} = \frac{\Sigma M_A \cdot \frac{l-y}{l} \cdot \frac{y}{i} \cdot dy}{\Sigma M_B \cdot \frac{y}{l} \cdot \frac{l-y}{i} \cdot dy} = \frac{\frac{\eta}{i_a} \Sigma \frac{M}{h} \cdot dy}{\frac{\eta'}{i_b} \Sigma \frac{M'}{h} \cdot dy}$$

Now to the base $\frac{l}{2}$

$$\Sigma \frac{M}{h} dy \quad \text{and} \quad \Sigma \frac{M'}{h} dy$$

are represented by

$$\frac{M_A}{h} \quad \text{and} \quad \frac{M_B}{h},$$

whence it is apparent that

$$\frac{\eta}{i_a} = \frac{\eta'}{i_b}.$$

From whence it follows that the line $A'B'$ also cuts the beam AB in a point in the line y , and y possesses the same properties as that of the earlier construction.

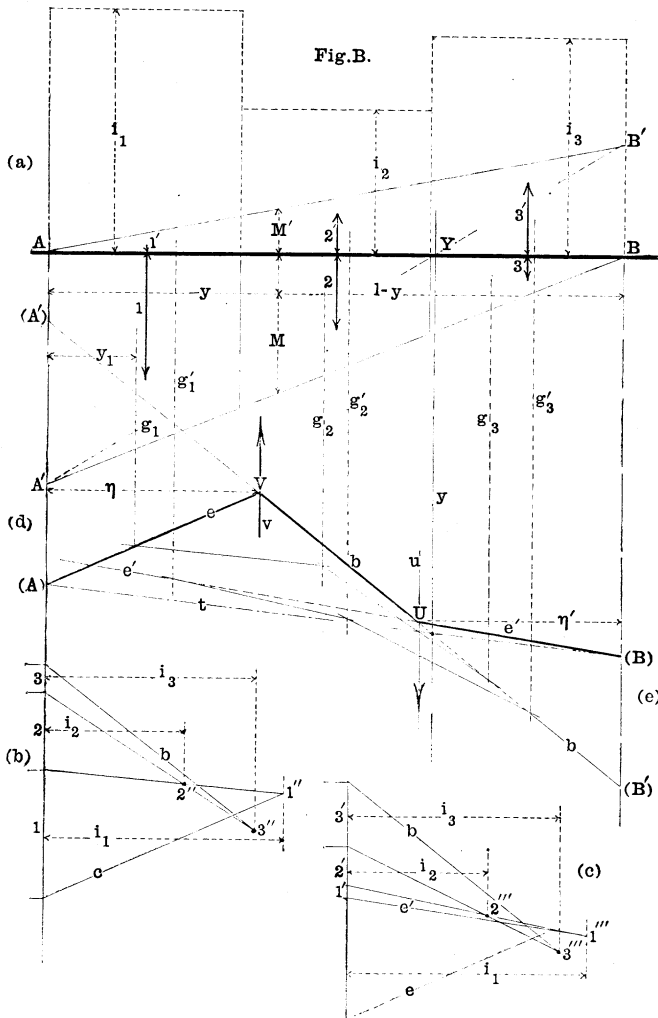
Having therefore found the lines u and v we require the lines w of the resultants of U and V

$$V' = \frac{1+2+3}{i_a}, \quad U' = \frac{1'+2'+3'}{i_b}$$

or

$$V' \cdot i_a = 1+2+3, \quad U' \cdot i_b = 1'+2'+3',$$

reduce $V' \cdot i_a$ and $U' \cdot i_b$ to some common base k , when the



resulting lengths V'' and U'' can be used to find the line w of the resultant W .

The construction of the elastic envelope and determination of the moments of a continuous beam with variable cross section can now be accomplished, with the help of the methods previously elucidated.

173. *Maxima of Shearing Force and Bending Moment in any Span of a Continuous Beam under Concentrated Loads.*

I. *Shearing Force.*—The same rules prevail for a positive or negative maximum of shearing force at any section as for a girder lying freely upon two supports.

II. *Bending Moments.*

a. *Investigation.*—Referring to fig. 123, page 217. Let AB as formerly = l .

$$AX = a$$

$$YB = b$$

$$AP = y$$

$$A \text{ to } g = \frac{l+y}{3} \text{ from elementary considerations.}$$

$$PH = Py \frac{l-y}{3} \text{ pole distance } h \text{ being regarded as unity.}$$

$$AA'' = \frac{Py(l-y)}{l} \cdot \frac{l+y}{l} = M_A + \frac{l-b}{b} \cdot M_B \quad \dots \quad (1)$$

$$BB'' = \frac{Py(l-y)}{l} \cdot \frac{2l-y}{l} = M_B + \frac{l-a}{a} M_A \quad \dots \quad (2)$$

from (1) and (2) we obtain

$$M_A = \frac{Py(l-y)}{l} \cdot \frac{a}{l} \cdot \frac{2l-3b-y}{l-a-b} \quad \dots \quad (3)$$

$$M_B = \frac{Py(l-y)}{l} \cdot \frac{b}{l} \cdot \frac{l-3a+y}{l-a-b} \quad \dots \quad (4)^1$$

M_A is a maximum when

$$y = l - b - \sqrt{\left\{ \frac{l^2}{12} + \left(\frac{l}{2} - b \right)^2 \right\}} \quad \dots \quad (5)$$

¹ Equations 1 to 4 are easily obtainable. The two former by means of similar triangles; the two latter by the most ordinary algebraical operations, so that it has not been considered necessary to retain them.

M_B is a maximum when

$$y = a + \sqrt{\left\{ \frac{l^2}{12} + \left(\frac{l}{2} - a \right)^2 \right\}}, \quad \dots \quad (6)$$

whence, a force applied at these points generates the greatest bending moment over the A and B supports.

b. Construction. These values of y are easily constructed. In fig. C ,

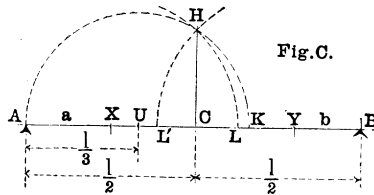


Fig. C.

$$AC = BC = \frac{l}{2},$$

$$UA = UH = UK = \frac{l}{3},$$

$$AX = a,$$

$$BY = b,$$

$$XH = XL,$$

$$YH = YL',$$

L and L' are the points sought.

In this chapter we have not attempted to give a practical example, for, were we to employ very few plates for the purpose, they would be so crowded with lines as to be almost unintelligible; and to employ many plates would go beyond the limits of this work. But the careful student may from the foregoing paragraphs fully work out the whole of any example.

¹ Freeing equation (3) of constant factors we have

$$y(l - y)(2l - 3b - y) = (2l^2 - 3bl)y + (3b - 3l)y^2 + y^3.$$

For a maximum we differentiate and equate to zero.

$$\frac{d}{dy} \left\{ (2l^2 - 3bl)y + (3b - 3l)y^2 + y^3 \right\} = 0,$$

$$\frac{2}{3}l^2 - bl + 2(b - l)y + y^2 = 0,$$

a quadratic equation giving the expression (5) in the text. A similar proceeding gives expression (6).

CHAPTER V.

THE ARCH.

Section 1.—On Pressures Communicated through Bodies in Contact with Each Other.

Fig.134.

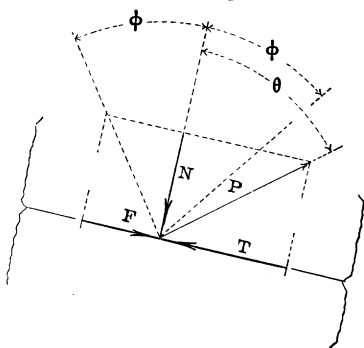


Fig.133.

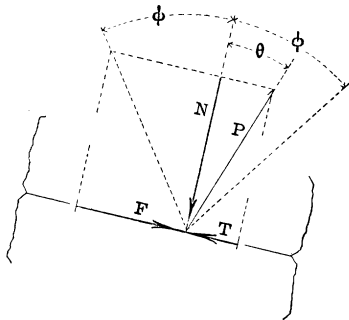
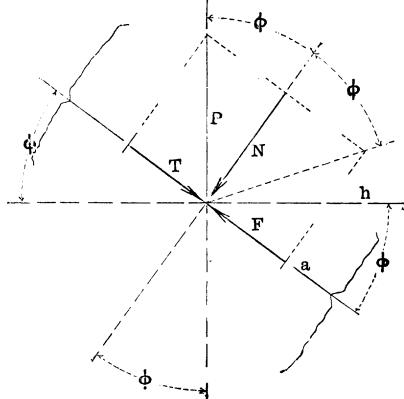


Fig.135.



174. *Value of Friction in Terms of Pressure, and of Stability in reference to Sliding, figs. 133, 134, 135.*—The well-known equations of friction (1 and 2) between the surfaces in contact of two solid bodies, the one being immovable and the other in a position to slide upon the first by means of a suitable pressure P , is

obtained by resolving P into two forces, one N , normal to the surfaces in contact, and one T tangential to them. The normal pressure N causes contact between the surfaces, the tangential pressure T tends to cause sliding between them, but sliding will not take place when the friction F is greater than T , *i.e.* when

$$F > T \text{ (fig. 133)}$$

and

$$F = fN. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let P make an angle θ with the normal to the surfaces in contact, then

$$N = P \cdot \cos \theta$$

$$T = P \cdot \sin \theta$$

and

$$F = fN = fP \cos \theta$$

so, in order that sliding may not occur, we must have

$$fP \cos \theta > P \sin \theta \text{ (fig. 134),}$$

or

$$f > \frac{\sin \theta}{\cos \theta} > \tan \theta,$$

and by parity of reasoning for sliding to occur

$$f < \tan \theta.$$

There must be a value of θ , different for different substances in which the action of P borders upon originating a sliding motion. This angle is usually denominated by ϕ , and we have

$$f = \tan \phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

whence it follows that sliding action is independent of P .

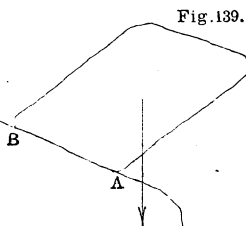
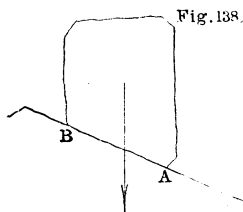
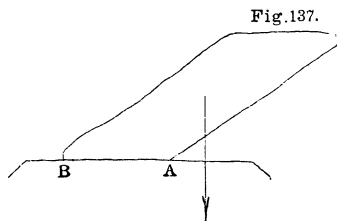
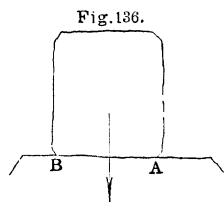
The average value of ϕ in stone joints is about 35° , whence in order that sliding do not take place in a masonry joint, the direction of pressure at the surfaces in contact must not deviate from the normal more than 35° .

175. *Plane of Repose.*—Consider fig. 135. The pressure P is vertical and might be due to the weight of the upper body. The surfaces in contact of the two bodies are inclined at the angle ϕ to the horizontal. It is evident that in this case the

normal pressure N makes the angle ϕ with P , whence (158) $F = T$, and the upper body is in a state bordering upon motion.

In this case ϕ is called the angle of repose.

The two bodies in contact may be two voissiors of an incomplete arch, or they may be two prisms of earth separated by an ideal plane, making an angle ϕ with the horizontal, the upper prism is then in a state bordering upon sliding motion over the lower prism, and the ideal plane is called the plane of repose.



176. *Stability in Reference to Overturning.*—The well-known condition that a body placed upon a plane surface stand or fall, is, that the resultant of pressures applied to that body go through the surfaces in contact.

The simplest case of that condition is, when the pressure applied to a body is its own weight going through its centre of gravity.

Let figs. 136, 137, be two vertical sections of two bodies resting on a horizontal plane a , let figs. 138, 139, be two bodies resting on an inclined plane a' but prevented from slipping by friction, *i.e.*

$$P \wedge N = \theta < \phi \text{ or } a \wedge h = \theta < \phi,$$

then in figs. 136, 138, the weight cannot overturn the stone around either A or B , because its effect is destroyed by the reaction of the plane a , but in figs. 137, 139, the weights have a moment around A , and the stones must be overturned.

177. *Line of Pressure in a Series of Bodies in Contact with One Initial Impressed Force R*, fig. 140.—The plane upon which a stone is placed may be a joint a in an uncemented masonry wall, and the initial pressure P , the resultant R of all the pressures, acting upon it, however originated, and coupling it with the weight A of the stone A acting vertically through its centre of gravity A , we obtain the direction $A'B'$ of the resultant pressure. In the same manner, by means of force and cord polygon, we obtain the links $B'C'$, $C'D'$, $D'E'$ of the cord polygon, necessarily the resultant direction of the pressures arising from successively adding the weights B, C, D of the stones B, C, D These links form the line of pressures A', B', C', D'

178. MOSELEY'S¹ *Line of Resistance in a Series of Bodies in Contact*.—Extending the ray $A'B'$ back till it intersects the joint a , we obtain a point a in that joint in which the resultant of R and A acts, whence (176), when this point is within the joint the stone A will not be overturned. a is said to be a point in the line of resistance. Extending the rays $B'C'$, $C'D'$, $D'E'$ back to the joints b, c, d we obtain further the points b, c, d in the line of resistance. Joining them, we have the line of resistance. As these points in our figure fall inside of the joints it represents a stable system.

In fig. 140 the vertical through A meets R below the joint a , the vertical through B meets $A'B'$ below the joint b , the vertical through C meets $B'C'$ below the joint c and the line of resistance in consequence falls wholly on the superior side of the line of pressures. But if the verticals through A, B, C had met $RA', A'B', B'C'$ within the joints a, b, c, d, e then the lines of pressure and resistance would have formed two polygons, the former circumscribing the latter. This the

¹ *Mechanical Principles of Engineering*. Lond. 1855.

student might verify for himself by taking the data of fig. 140, and reconstructing it with an initial pressure R more nearly horizontal.

The line of resistance a, b, c , is not drawn, as, for these joints, it almost coincides with the line of pressures, and would be a little confusing in the figure.

Section II.—The Arch considered.

179. *Lines of Pressure and Resistance in an Arch, fig. 141.*—Let a semi-arch be supported at its highest joint by a pressure H . The line of resistance deviates but little from the line of pressures. In order to render the deviation sensible we must take a semi-arch with exceedingly long voussoirs near the springing where the deviation increases most rapidly. Fig. 140 renders this evident.

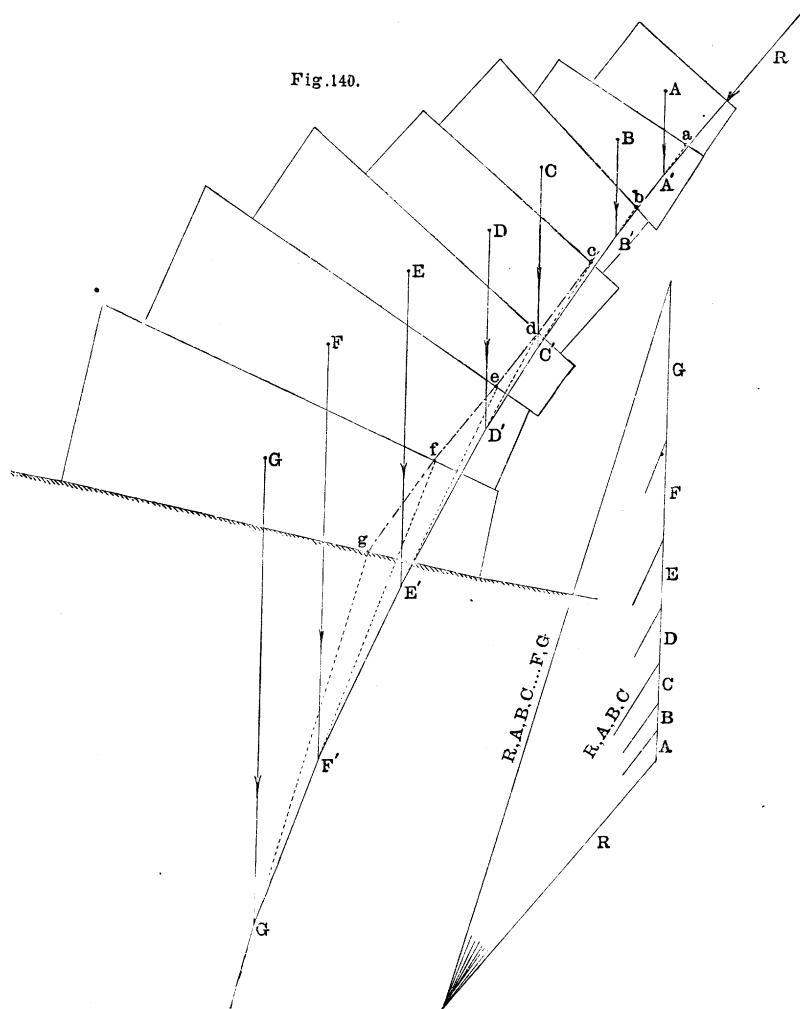
If the voussoirs of an arch are supposed to be indefinitely narrow vertical laminæ as Δ (fig. 141), the line of resistance coincides with the line of pressures. For the verticals through their centres of gravity must necessarily meet the cord polygon links within the joints of the laminæ, whence the two lines form two polygons of indefinitely small sides, the one circumscribing the other.

If the line of pressure is constructed on the supposition of vertical joints of a breadth which allows the portions of the arch intercepted by them to be treated as a straight line, as Δ' , it is much more easily obtained owing to the simplicity with which the centres of gravity of the vertical laminæ can be found.

180. *Recovering a Point in the Line of Resistance from the Line of Pressures.*—The method of obtaining the line of pressures in 163 is practically accurate for the line of resistance for a considerable distance from the summit of the arch, whilst a point in the line of resistance, at any oblique joint, can be found as following.

Let AB (fig. 141) be the imaginary vertical joint to which the coinciding lines of pressure and resistance have been carried, whereas ACD is the real joint, then through the centre of gravity of the figure $ABCD$, draw a vertical P cutting the direction Q of the pressure on AB . From this point (PQ) and along

Fig.140.



Q measure off its value, as obtained from the force polygon, and from its extremity lay off vertically upwards the line P , representative of the weights of $ABCD$, and complete the triangle of forces by Q' cutting AC in J . J is a point in the line of resistance. This proposition requires no formal demonstration.

In arches of considerable curvature it is of importance to obtain that point in the line of resistance where it cuts the springing, in order to obtain the true line of resistance in the abutment. For while the line of resistance is generally more favourable to the stability of the arch than the line of pressures, it is less favourable to the stability of the sustaining abutment.

181. *The True Line of Pressures is that which is nearest the Axial Line.*—Let the material of which the arch is composed be at first so weak that on account of this weakness the pressure line must be so far limited to the interior of the arch as to be capable of sustaining the reaction at Q and Q' ,¹ Q being in the axial line of summit, Q' in the axial line at springing, and consequently only one pressure line possible, then this line would be the pressure line for that weak arch. Now let the voussoirs be hardened without any other properties of the stones being altered, then this hardening cannot alter the position of the line of pressures, yet there can now exist other lines of pressure. The line of pressures is the linear arch of some writers.

182. M. DURAND CLAYE'S² *Method of Finding the Admissible Lines of Pressure in an Arch.*—This elegant method will be best unfolded by taking a definite example.

Fig. 142, is portion of a symmetrical stone arch of which the vertical 1. 2. 3. 4 is at the crown. This figure suffices to elucidate the method, although we do not take any loading into consideration, but did we include any, it would require to be symmetrically placed around the vertical through the crown.

By means of a force and cord polygon, we find the vertical \bar{a} through the centre of gravity of the portion in fig. 142.

From the points 1 and 4 in the vertical through the crown, draw in the two portions b of the equilateral hyperbolas of

¹ The reader will bear in mind that the pressure upon any layer of the arch is greater the further the resultant is from G .

² See note, page 144.

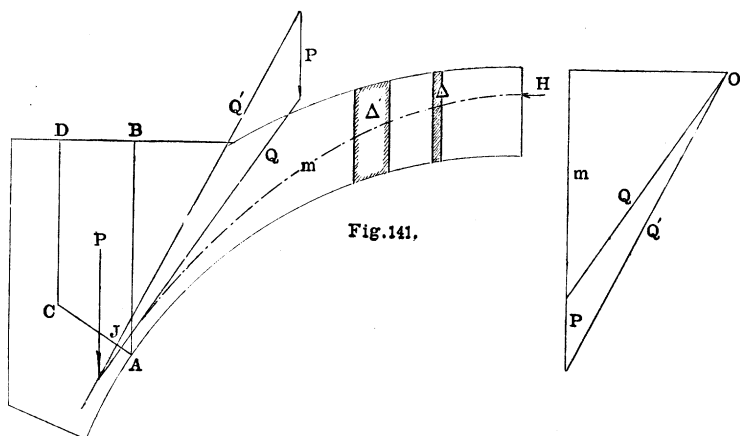


Fig. 141.

art. 122. These curves having been drawn in on the supposition that the material is granite having a resistance to crushing of 5000 lbs. per square inch. The thickness of the arch taken being 1", a resistance exceeding many times that which the largest bridges call into action, so that with an appropriate scale we can only find place in our figure for a portion of the tangential lines belonging to the outer thirds of the voussoirs.

For a number of points as 1. 2. 3. 4 within the vertical joint through the crown, draw tangents 1, $\bar{1}$, 2, $\bar{2}$, 3, $\bar{3}$, 4, $\bar{4}$. . . to the curvature at these points cutting the vertical \bar{a} through the centre of gravity. Take corresponding points 1'. 2'. 3'. 4' in the radial boundary of the last voussoir. In the vertical \bar{a} lay off $\bar{1} \bar{1}$, $\bar{2} \bar{2}$, $\bar{3} \bar{3}$, $\bar{4} \bar{4}$. . . measuring to some chosen base the weight of the portion of the arch (and its loading if any) operated on.

Now, to fix the ideas, take some point as $\bar{3}$ in \bar{a} , and from it draw lines to the points 1, 2, 3, 4 . . . of the radial boundary line r . Of these let us attend to one, say $\bar{3} 4'$ cutting the horizontal through $\bar{3}$ in the point $4''$, $\bar{3} 4''$ is the horizontal force at 3 in the crown which will give a line of pressures passing through 3 at the crown and $4'$ at the radial boundary line. Transferring this distance reduced to the same base as the hyperbolic lines, to a horizontal from 3, we obtain a point in the line a_4 . Similar operations will give as many points as we will in the line a_4 . The ordinate x to the curve a_4 through

any point z in the crown vertical, measures the horizontal force which must be applied at that point z in order that the resulting line of pressures go through the point $4'$ in the radial joint r .

Similar operations will give as many points as we will in the lines a_4, a_3, a_2, a_1 , and their ordinates are interpreted in a similar way.

The inferior extremities of the lines a_1, a_2, a_3 are at infinity, which means that an infinite horizontal force would require to pass through the inferior portion of the vertical joint at the crown, in order to generate a line of pressures passing through the points $1' 2' 3' \dots$.

Still fixing our attention on the lines $3\bar{3}, \bar{3}4'$, we can easily show by a reference to the principles of the cord and force polygon, that $\bar{3}$ measures the horizontal force at the crown necessary to cause the pressure line to pass through $3'$ and $4'$. For the line $3\bar{3}$ from the crown is the extreme ray e of our cord polygon, and $\bar{3}4'$ is the extreme ray e' cutting in the resultant \bar{a} of the weights. $\bar{3}3''$ is the line of weights of our force polygon. The horizontal line $3''4$ is at once the pole distance and the ray e of the force polygon, and the pole distance is likewise the horizontal pressure.

Taking another joint s enclosing between s and the crown a portion b of the arch, and repeating the same operations, we obtain further lines b_1, b_2, b_3, b_4 .

Again, with the whole semiarch from the crown to a joint t , at which a rise of 12 feet and a span of 72 feet has been attained, we obtain the lines $c_1 c_2 c_3 c_4$ (fig. 144).

We can now, by inspection, find that area within which the extremities of the horizontal ordinates give admissible values to the horizontal pressure to be impressed upon the joint at the crown. They must be within the hyperbolic space, but this is so large as to be beyond all other admissible boundaries. They must be within the lines $a_2 a_3, b_2 b_3, c_2 c_3$, in order that they may give lines of pressure within the central third, or more generally, within the core of the cross section. This area is hatched in fig. 144.

The various values of the pressures, bounded for instance by

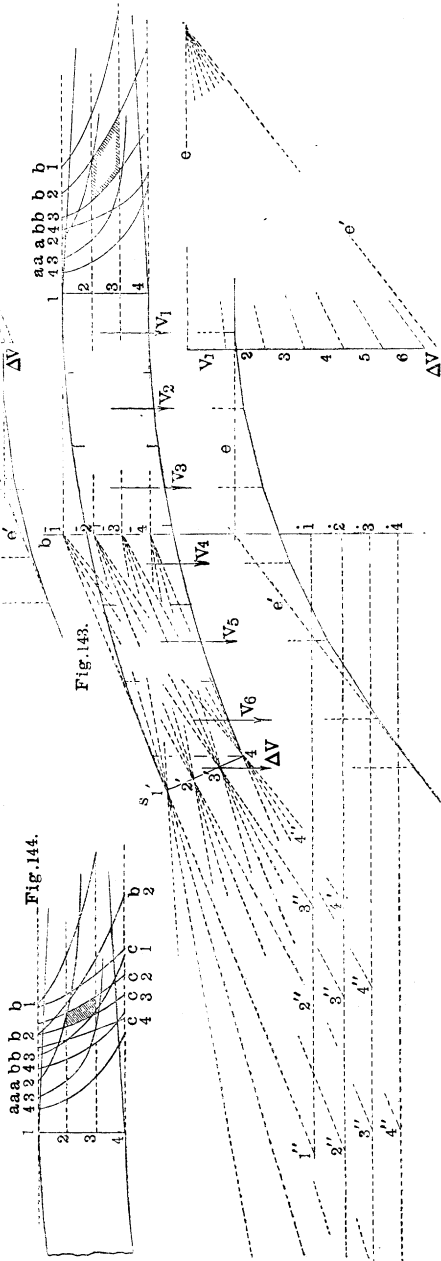
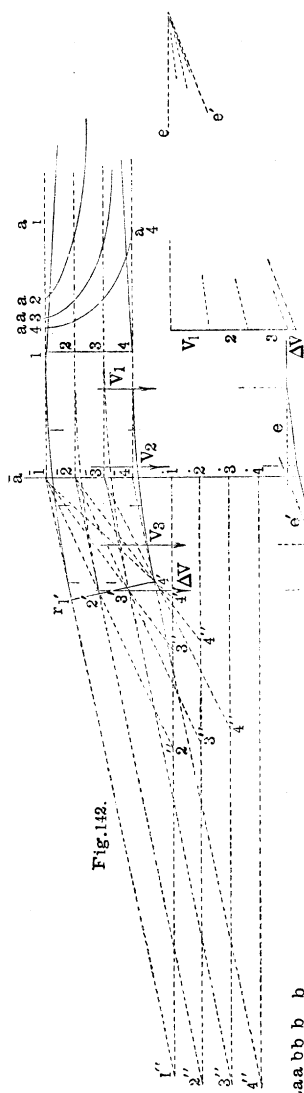
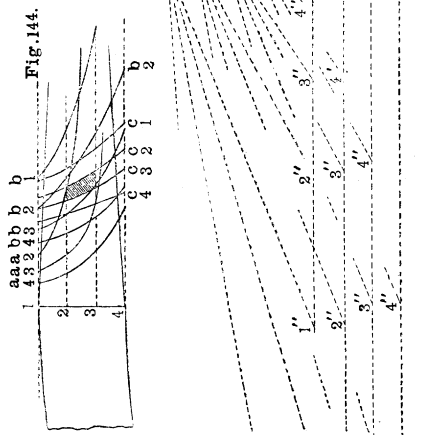


Fig. 144.

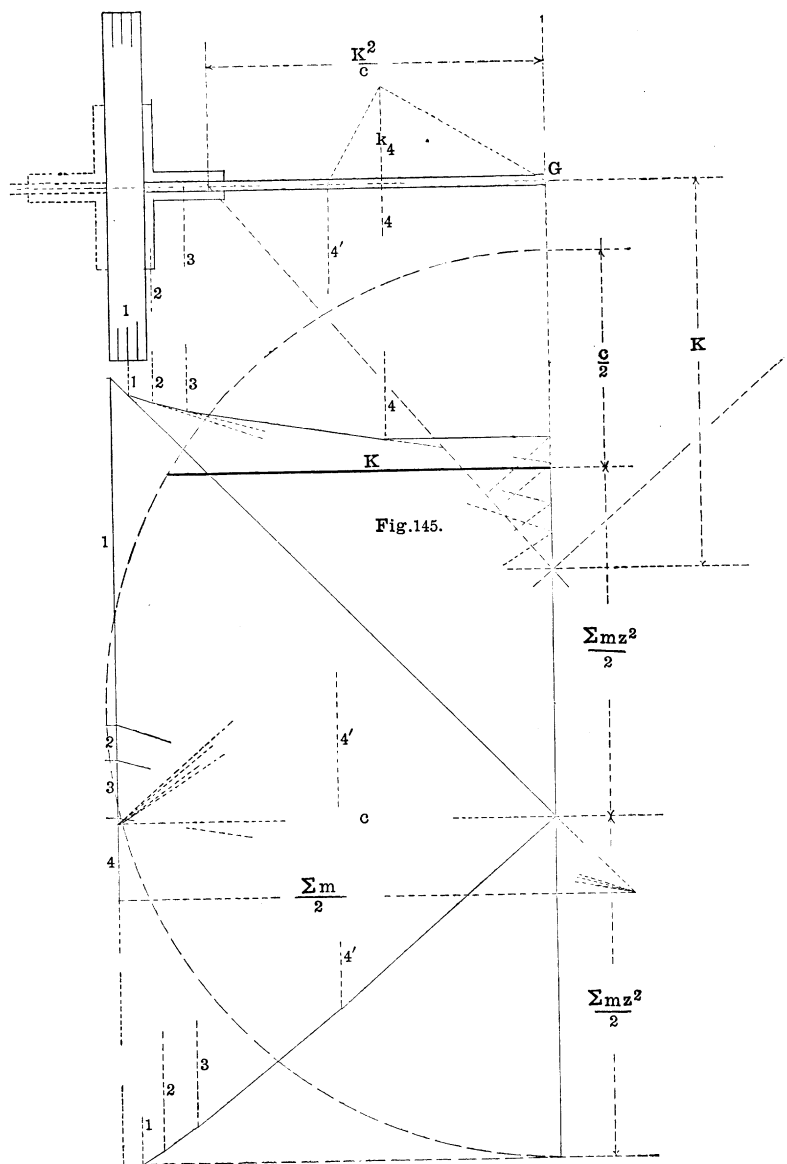


c_1 , give at t_1 a series of values, at t_2 another series of values, at t_3 another series and we thus find a kind of reciprocal limiting area at the joint t which may have the effect of further limiting the previously found area, but this is generally unlikely. (These t lines are not given.)

183. *Application of HEUSER'S Problem to Archwork.*—Heuser's problem (74) is a very efficient instrument for investigating the stresses in an arch, and can readily be applied to unsymmetrical loading and unsymmetrical forms.

Example.—We have already seen (114) that in order that there be no tension in an arch, the resultant stresses must pass within the two principal points of the core or heart. Fig. 145 is a provisional cross section of the arch in fig. 146, in which the operations necessary to find these two points are carried out (119) distant $\frac{K^2}{c}$ from G . Fig. 146 is the longitudinal section of a rib, showing only, however, the trace of these two points in two full but fine lines. The arch is divided into seventeen parts, each 10 feet long, except the seventeenth, which is 15 feet long. The first eleven parts have only the structural weight, the remaining parts are loaded with 2 tons per lineal foot. Taking, then, the lower point at the left abutment in the trace of the core, for the point A and the upper point at the right abutment in the trace of the core for the point B , and as the unloaded portion of the cord polygon is necessarily the flattest, exercising our judgment, we choose a point in the lower trace of the core between the structural weights 6 and 7 for the point C , then by means of Heuser's problem we take a pressure line through these three points. This line of pressures lies beyond the core on the upper side in the parts 13 and 14, thus introducing tension at that place on the inferior side of the rib. It is evident that we cannot lower the line of pressures further, without lowering it below C , and thus introducing there, tension on the superior side of the rib, whence it follows that in the region of 13 and 14 and the region symmetrical thereto, the arch must be strengthened both on upper and lower side.

184. *Curvature of Arch.*—The longitudinal axis traversing the centres of gravity of the cross sections, ought to coincide



with the line of pressures of its structural weight. In France¹ and England this has not been done, the curvature being invariably the segment of a circle, but in Germany the more correct treatment has been followed, as, for instance, over the Rhine at Coblenz.² Supposing this weight to be uniformly distributed, that axis would be a parabola, but should there be an important deviation from this condition, the true line of pressures may be obtained by the method of Chap. VI. for finding the form assumed by the chains of a suspension bridge. This method, besides its structural propriety, has the further advantage of allowing the investigation of the stresses arising from travelling load, to be conducted apart from the structural weight.

185. *Finding the Most Axial Line of Pressures for given Loading.*—In Fig. 146, the line of pressures at 13, 14, in the original drawing, are barely beyond the core. Supposing then for a moment that at that place it touches the exterior limit of the core, then at *C* it touches the interior limit, and again at *B* and *A* it does the same; we may reckon then that we have obtained for that loading the most axial line, for were we to depress our line at 13, 14, we should also depress it at *C*, but had it been within the core at 13, 14, we might elevate it within the core at both places, perhaps at *A* depress it, perhaps at *B*. In this manner, we can, *tentatively*, discover the most axial line for any loading. In order to do this, however, it is only necessary to operate with cord polygons of four sides passing through *three* given points as *A, B, C*, and determining a fourth as 13, 14. When once, however, it has been determined, it may be necessary to insert all the sides.

These lines of pressure require, in cases of any importance, to be determined for loadings over fractions of the span of 0·1, 0·2, 0·25 0·9, also for loadings distributed in the manner elucidated in art. 223, p. 284, and the maxima of bending and radial shearing stress determined.

186. *Bending and Radial Shearing Stress in Arch.*—The latter is the stress along the radius of curvature of the beam, the former is normal thereto (120), and are thus obtained :—

¹ Culmann, *Graphische Statik*, p. 597, Zürich, 1875.

² Hartwich, *Die Rheinbrücke bei Coblenz*, Berlin, 1869.

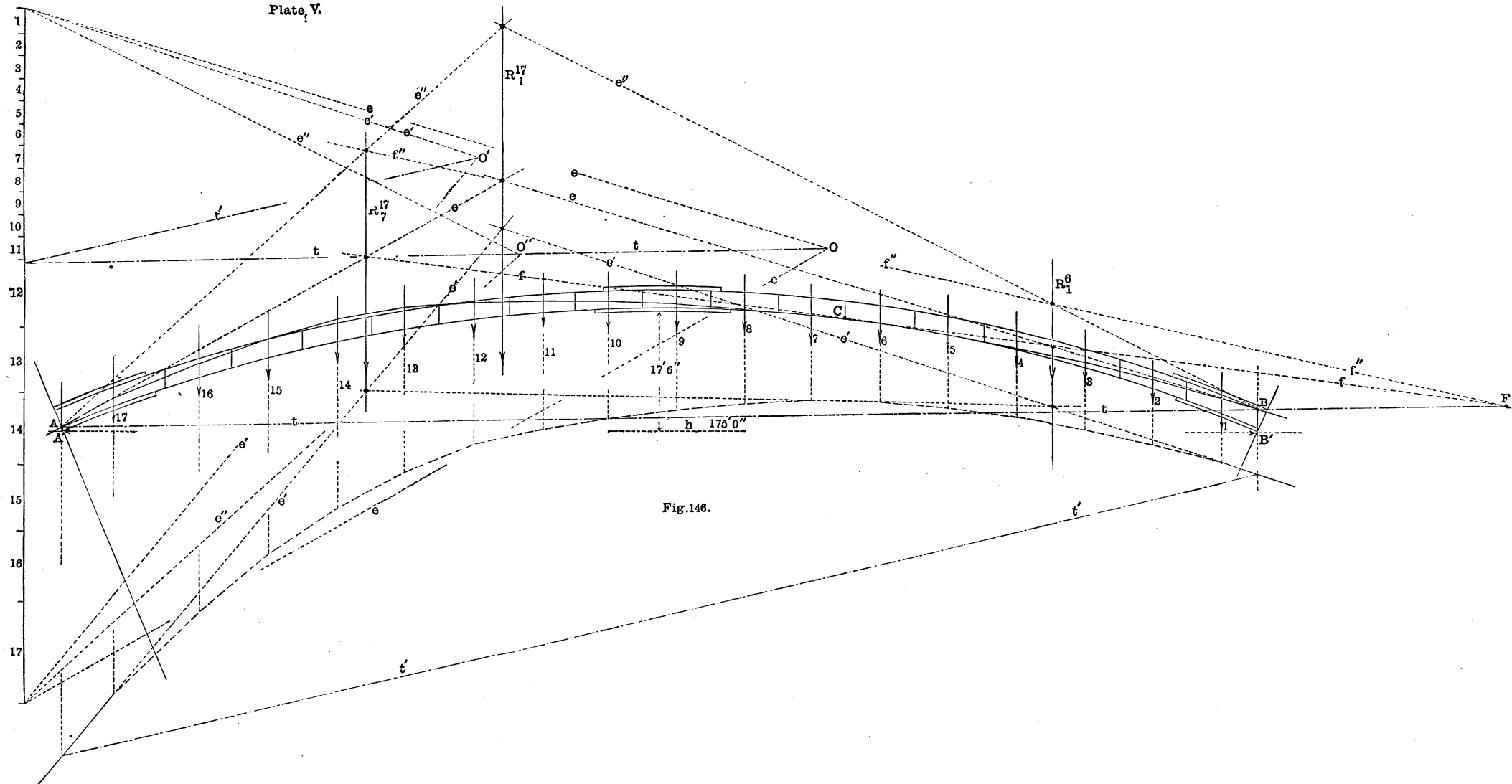


Fig. 146.

Fig.D.

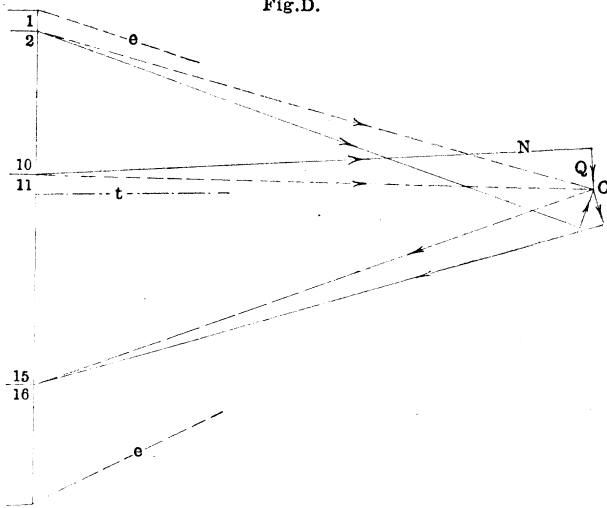
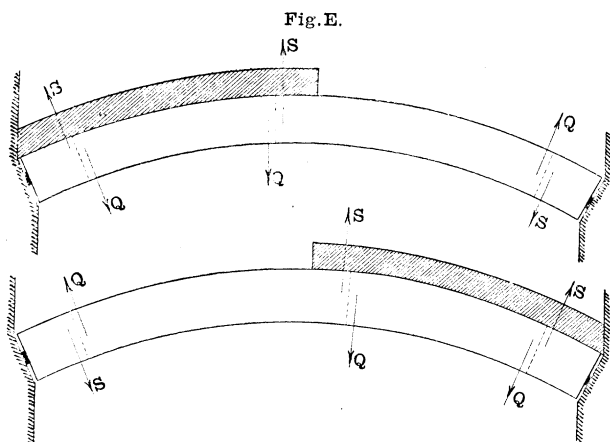
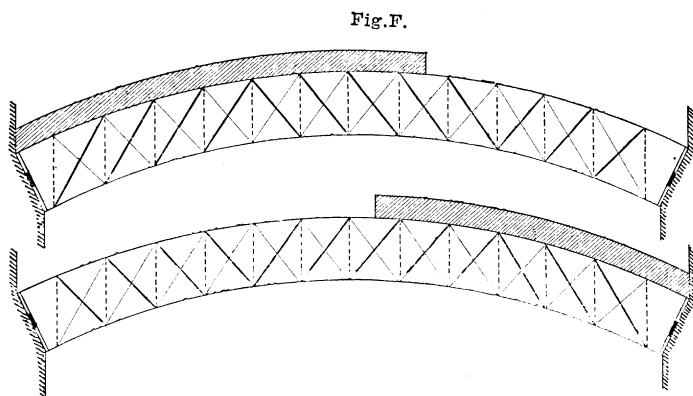


Fig. *D* is the force polygon of Fig. 146 to a scale of one half, in which if we mark the rays with arrows in the direction of the impressed forces, the rays above *t* become directed to the right abutment *B*, and those below to the left abutment *A*. Let us take, for example, the ray *O*(10, 11), resolving it into two forces *Q* and *N*, the former in the direction of the radius of curvature and the other normal thereto, and *substituting* them for the pressure along *O*(10, 11) we have *N* directed to the right and *Q* directed downward, *N* is the normal or bending force, *Q* the shearing force resisted by stress *S* acting upwards. Taking again the ray *O*(1,2) we find *Q* directed upwards, and *S* consequently acts downwards. At ray *O*(15,16) we find *Q* again downwards, whence we perceive that *Q* changes its sign between the superior *e* ray and *t* passing through zero, again changing its sign between *t* and the inferior *e* ray. We can thus obtain the value of *Q* and *N* for every ray of the force polygon. A general view of the changes of sign which *Q* undergoes is given in fig. *E*.

We need scarcely observe that in order to find the maxima values, it is most convenient to lay them off as ordinates to various curves, from a straight line, the development of the axis of the arch, when the maxima values will give points in an enveloping curve.



187. *Stresses in Tables of Beam.*—The stresses in the tables of the arch beam can be found from N by the assistance of the geometrical construction of Durand Claye's hyperbolas.



188. *Stresses in Bars of Open Arch Beam.*—The effect of the change of sign in Q upon the bars of an open arch beam is shown in fig. F , where thick lines denote compression, and thin lines denote tension, and from which we perceive that all the bars of an open arch undergo both compression and tension.

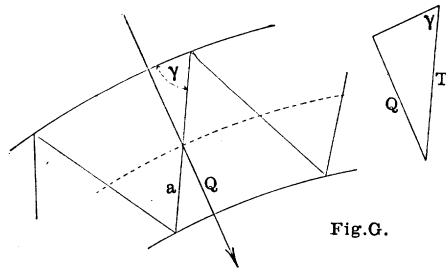


Fig. G.

In fig. *G*, let stress along the bar *a* be denoted by *T*, then

$$T = Q \cdot \operatorname{cosec} \gamma \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence the simple construction of *T* in the figure, and which represents a ^{tension} according as *Q* acts ^{upward} _{compression} ^{downward}.

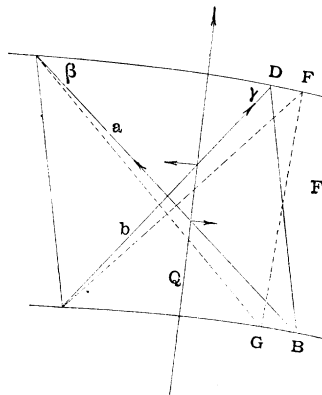


Fig. H.

In the case of bars in the form of a St. Andrew's Cross (fig. *H*), let the bar rising from ^{right} to ^{left} be denoted by ^{*a*}/_{*b*}, β and γ their angles with the superior table, then

$$Q = T_a \sin \beta + T_b \sin \gamma \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let the bars, owing to the bending of the arch under stress assume the positions shown by the dotted lines, we have approximately

$$DF = BG = \sigma \quad . \quad . \quad . \quad . \quad . \quad (3)$$

As the stresses of the bars are proportional to the alterations in their lengths, we have

$$\frac{T_a}{E} = \frac{\sigma \cdot \cos \beta}{a} \text{ and } \frac{T_b}{E} = \frac{\sigma \cdot \cos \gamma}{b} \quad (4) \text{ and } (5)$$

wherefore through division

$$T_b = T_a \cdot \frac{\cos \gamma}{\cos \beta} \cdot \frac{a}{b} \quad (6)$$

substituting in equa. (2) we have

$$T_a = Q \cdot \frac{2b \cos \beta}{b \sin 2\beta + a \sin 2\gamma} \quad (7)$$

$$T_b = Q \cdot \frac{2a \cos \gamma}{b \sin 2\beta + a \sin 2\gamma} \quad (8)^1$$

The above operations are tedious, but in situations where economy and elegance are both required, an arch well recompenses all the care which can be bestowed on its design.

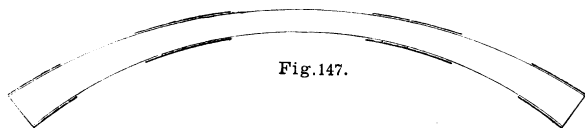


Fig. 147.

189. *Critical Sections of an Arch.*—Fig. 147 gives a general view of the critical parts of an arch; the parts shown in thick lines will be found to require more metal than the other parts.

190. *Conditions necessary to Equilibrium in a Pointed Arch.*—Let ACB (fig. 148) be the most axial line of pressure in a pointed arch. It is evident that we must have a heavy load at C in order to deflect it so abruptly, for the student is aware that the line of pressure is only our cord polygon, and that this load at C must be equal to that which would have been distributed over the extension of the two arcs AC , BC to the horizontal, is evident from a moment's consideration of fig. 17, and of the force polygon of the pressure line. A formal demonstration may therefore be omitted.

¹ Heinzerling, *Die Eisernen Bogenbrücken*, Aachen, 1880.

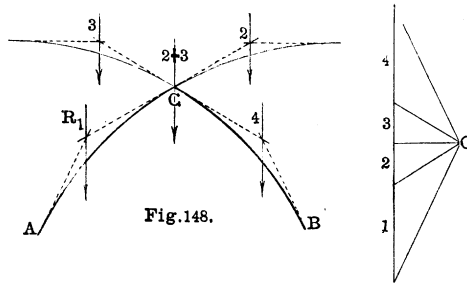
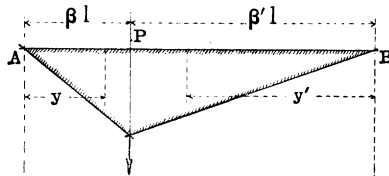


Fig. 148.

Section III.—The Elastic Arch. Professor CULMANN'S Method.

Part A.—The Equations of the Elastic Arch.

Fig. 1.



191. Lemma. Expression for the Bending Moment of a Force upon a Beam having Two Points of Support at any Section of the Beam.—Let fig. 1

APB be a beam,

$AB = l$, the span,

P an impressed force,

$AP = \beta l$, $PB = \beta' l$, (whence $\beta + \beta' = 1$),

y , the distance of any section from A , between A and P ,

y' , " " " " B , " B " P ,

then the reactions at A and B are known to be

$$A = P\beta',$$

$$B = P\beta,$$

also, moment of P in line of its action,

$$A\beta l = P\beta'\beta l,$$

whence, moment of P at any section y' between B and P is given from the following proportion:—

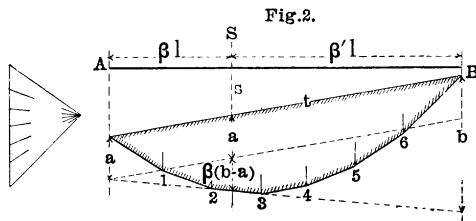
$$\beta'l : P\beta'l :: y' : P\beta y'$$

or

$$M' = P\beta y'.$$

In the same manner, the moment of P at any point y between A and P is

$$M = P\beta'y.$$



192. *Lemma. Expression for the Bending Moment Ordinate of a Beam, in a Cord Polygon, in terms of the Intercepts upon the Two Verticals through the Points of Support cut off by the sides of the Polygon bounding the Ordinate.*—Let it be required to determine, in these terms, the bending moment ordinate on the line s , fig. 2, bounded by sides t and $\overline{2, 3}$ of the cord polygon. Extend the side $\overline{2, 3}$ till it cuts the verticals through the points of support. Call the intercepts cut off between t and $\overline{2, 3}$ upon these verticals, respectively a and b , of which a is the least. Through the extremity of a draw a line parallel to t . Let

$$l = \text{span,}$$

$$AS = \beta l, SB = \beta' l, (\beta + \beta' = 1),$$

then the whole ordinate in question is

$$a + \beta(b - a).$$

For

$$l : b - a :: \beta l : \beta(b - a).$$

Again

$$\begin{aligned} a + \beta(b - a) &= a + \beta b - \beta a \\ &= a(1 - \beta) + \beta b \\ &= a\beta' + b\beta. \end{aligned}$$

193. *Differential Equation of Form Alteration in the Elastic Arch.*

i. *Symbols.*—In references to art. 101, let it be understood that

| s of that article is the | | | | ds of this chapter | | |
|--------------------------|---|---|---|-----------------------|---|---|
| ds | " | " | " | dφ | " | " |
| Zy | " | " | " | M | " | " |
| I or Σmz ² | " | " | " | I or Σmz ² | " | " |

ii. *Alteration in the Elementary Portion ds of the Arch, whose Length ds is measured on the Longitudinal Axis.*—Let R be the resultant of the forces acting on an element ds , figs. 3 and 4, where figs. ³/₄ represent the force as acting in a line within the core of the section, then we can resolve it into two forces Q and S of which Q alone is a bending force, acting with lever arm q , so that

$$Qq = M \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and S is simply a shearing force.

Q also gives a force

$$\frac{Q}{A} = p_y = p_z$$

distributed over the whole area A of the cross section.

Now (equa. 8, art. 96)

$$p : E :: d\phi : ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same manner

$$p_y \text{ or } p_z : E :: z d\phi : ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (2a)$$

and (equa. 10, art. 101)

$$p = \frac{M}{I},$$

whence

$$\frac{M}{I} : E :: d\phi : ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or

$$d\phi = \frac{Mds}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

194. *Turning Power of Elementary Form Alteration.*—Equation (4) gives the elementary turning of ds around the neutral axis of the cross section. From the point in the longitudinal axis around which Q turns, draw a straight r to the end A of this axis, it follows that r turns round an angle $d\phi$ and consequently the end A of the axis is displaced by a quantity $rd\phi$.

195. *Elementary Turning Power referred to Coordinate Axes.*—Let rectangular coordinate axes be taken through A as origin, fig. 5, then let the turning $rd\phi$ at A referred to these coordinates be dy and dz , and from (4)

$$\left. \begin{aligned} dy &= z \cdot \frac{M}{EI} \cdot ds \\ dz &= y \cdot \frac{M}{EI} \cdot ds \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

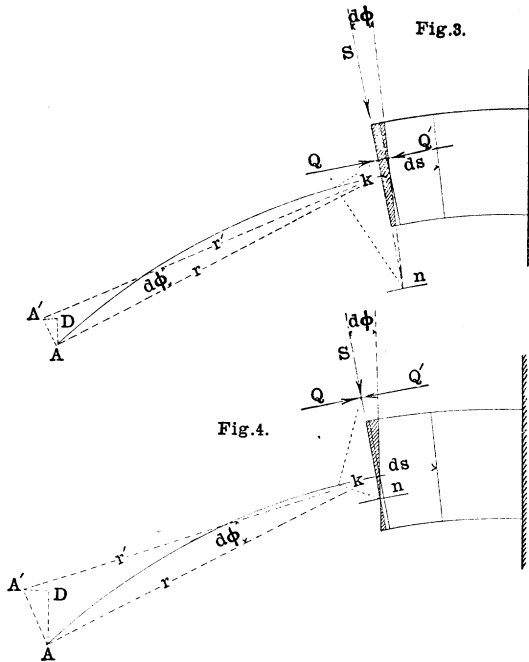
196. *Summation of Elementary Turnings.*—Every ds contributes an increment to the total turning. Let y'' , z'' be the ordinates of the pivot of the resultant turning, arising from all the increments.

Let

$$\left. \begin{aligned} \Sigma d\phi &= \phi = \Sigma M \frac{ds}{EI} = \phi \\ \Sigma dy &= \eta = \Sigma z M \frac{ds}{EI} = z'' \phi \\ \Sigma dz &= \zeta = \Sigma y M \frac{ds}{EI} = y'' \phi \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

197. *Hypothesis of a Single Force U Equilibrating the Whole Turning.*—Let us suppose that the sum of the effect produced by the moment M is equilibrated by a single force U acting

with the variable leverage u upon all points of the longitudinal axis of the arch, then substituting in (6) we have



$$\left. \begin{aligned} \phi &= \Sigma M \frac{ds}{EI} = U \Sigma u \frac{ds}{EI} \\ \eta &= z'' \phi = \Sigma z M \frac{ds}{EI} = U \Sigma zu \frac{ds}{EI} \\ \zeta &= y'' \phi = \Sigma y M \frac{ds}{EI} = U \Sigma yu \frac{ds}{EI} \end{aligned} \right\} \dots (7)$$

198. *Treating all $\frac{ds}{EI}$ as Forces, y'' and z'' , the Ordinates of the Pivot of Resultant Turning are the Ordinates of the Antipole V of the Line of Action of U , in reference to the Central Ellipse of all $\frac{ds}{EI}$.*

i. *Symbols.*—Let the value and line of action of U be considered as known.

Let centre of gravity of $\Sigma \frac{ds}{EI}$ be G .

$\Sigma \frac{ds}{EI}$ be denoted by s .

u be any ordinate from U to a corresponding $\frac{ds}{EI}$.

u_g the ordinate from U to G .

In reference to the central ellipse of $\Sigma \frac{ds}{EI}$.

Let Y be the antipole of the axis y .

Z the antipole of the axis z .

y_g, z_g , the coordinates of G to the axes z and y .

y_u, z_u , the coordinates of V the antipole of U to the axis z and y .

u_y, u_z , the ordinates from U of the antipoles Y and Z , then

ii. Equation 7, becomes

$$\phi = U \cdot \Sigma u \frac{ds}{EI} = U u_g s \quad \dots \quad (8_1)$$

From art. 131 iv. we may write

$\Sigma uz = \Sigma$ ordinates from axis $U \times$ ordinates from axis y .

= Central ordinate from axis $U \times$ antipolar ordinate of axis U measured from axis y .

= Central ordinate from axis $y \times$ antipolar ordinate of axis y measured from axis U .

or

$$\Sigma uz = u_g z_u = z_g u_y,$$

also

$\Sigma uy = \Sigma$ ordinates from axis $U \times$ ordinates from axis z .

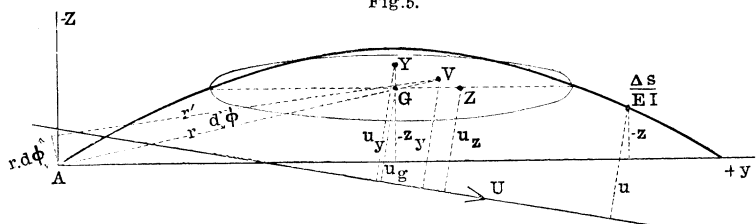
= central ordinate from axis $U \times$ antipolar ordinate of axis U measured from axis z .

= central ordinate from axis $z \times$ antipolar ordinate of axis z measured from axis U ,

or

$$\Sigma uy = u_y y_u = y_g u_z,$$

Fig. 5.



whence, equations 7₂, 7₃, and substitution from 8, give

$$z''\phi = U\Sigma uz \frac{ds}{EI} = U\Sigma u_g z_u = U\Sigma z_g u_y = \phi z_u \quad (8_2)$$

$$y''\phi = U\Sigma uy \frac{ds}{EI} = U\Sigma u_g y_u = U\Sigma y_g u_z = \phi y_u \quad (8_3)$$

whence

$$y'' = y_u, \quad z'' = z_u,$$

whence we have found that the axis of the arch turns round the antipole of the line of action of U .

199. *Value of Single Force U resisting Turning.*—From equation (8₁) we obtain

$$U = \frac{\phi}{u_g s} \quad (9)$$

200. *Multiples of Perpendiculars upon U from Centre of Gravity G and from the Antipoles Y and Z of y and z .*—Equations (8) involve the perpendiculars u_g, u_y, u_z from G, Y, Z upon U . They furnish likewise, the following relation

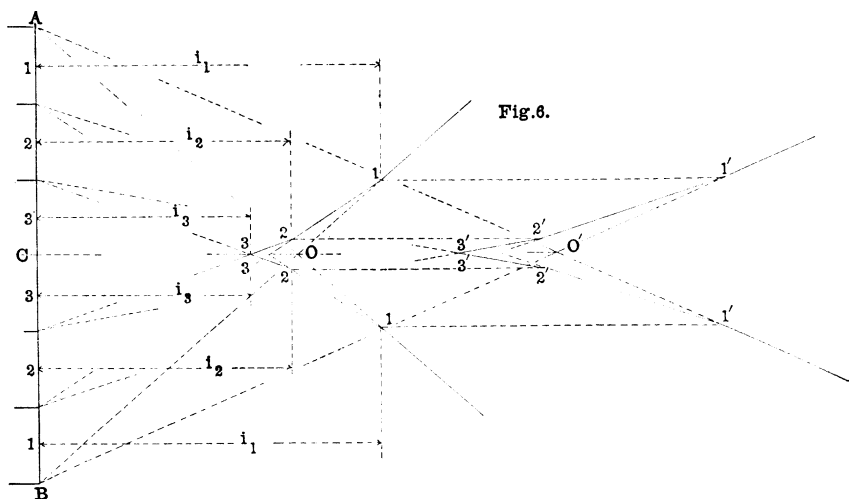
$$\frac{1}{Us} = \frac{u_g}{\phi} = \frac{u_y}{\eta} = \frac{u_z}{\xi} \quad (10)$$

The numerators u_g, u_y, u_z of these equal fractions, are the perpendiculars from G, Y, Z , upon U and this equation shows that

$$\phi, \frac{\eta}{z_g}, \frac{\xi}{y_g}$$

are respectively proportional to them, so that if we obtain the values of these denominators, we can construct the line of action and value of force U .

Section III.—Part B. Graphical Construction of the Lines of Action and Values of Reacting Forces.



201. *Preliminary Problem.*—In order to avoid confusion in the large figure in which the following constructions are carried out, we have represented in figure 6, a force polygon of the class of fig. *B, c, d*, of page 243, but in which the parts of the line of weights are symmetrical, and the moment of inertia intercepts i , are symmetrical with regard to the middle of the line of weights. Its construction giving the pole O is manifest. It is now required to transform it so that it may have a *given* pole distance CO' . For this we must have

$$\frac{CO}{CO'} = \frac{i_1}{i_1 + 1, 1'} = \frac{i_2}{i_2 + 2, 2'} = \dots$$

This the construction of the figure gives, it is evidently a simple case of homology.

In forming the first polygon, it is highly convenient to begin with $\overline{U, 3}$ perpendicular to AB , AC being $= BC$.

202. *Modification of the Form of the Increment $\frac{ds}{EI}$ or, employing Finite Increments, of $\frac{\Delta s}{EI}$.*

$$\frac{\Delta s}{EI} = \frac{\Delta s \frac{\Delta y}{\Delta s}}{EI \frac{\Delta y}{\Delta s}} = \frac{\Delta y}{EI \frac{\Delta y}{\Delta s}} = \frac{\Delta y}{Eabc \frac{\Delta y}{\Delta s} i} \quad \dots \quad (11)$$

where a , b , c , and i have the signification of earlier articles and having obtained the intercept i of the second cord polygon, we multiply it by

$$\frac{\Delta y}{\Delta s},$$

or what is the same thing, we obtain

$$\frac{\Delta y}{\Delta s} i$$

by employing as a base not c , but

$$c \frac{\Delta s}{\Delta y}.$$

Divide the orthographic projection AB of the bow, plate V.*a*, upon the line of abscissae into an arbitrary number of equal parts (fourteen in this figure). These are consequently projections Δy of unequal lengths Δs of the bow, for the cross section of each of which Δs parts, i has been obtained, and must now be multiplied by

$$\frac{\Delta y}{\Delta s}.$$

203. *Equating Sum of Moments in Bow with Moments of U .*—We will now put the sum of moments in bow under a form derived from lemma (191) with the sum of moments of resisting force U as given in equations (8), and instead of the earlier value of s in (198, i.), let us now take a multiple of it, so that now

$$s = \Sigma \frac{\Delta y}{\Delta s} i, \quad \dots \quad (12)$$

For the sake of brevity we will in future denote $\frac{\Delta y}{\Delta s}i$ by i only.

Let the moments in the bow originate from one force P acting vertically upon it at a distance y from A , then we have

$$\left. \begin{aligned} P\beta'\sum_0^{\beta l} y \frac{\Delta y}{i} + P\beta\sum_{\beta l}^l y' \frac{\Delta y}{i} &= U\Sigma u \frac{\Delta y}{i} = Uu_g s = \phi \\ P\beta'\sum_0^{\beta l} yz \frac{\Delta y}{i} + P\beta\sum_{\beta l}^l y'z \frac{\Delta y}{i} &= Uu_y z_u s = Uz_y u_y s = \phi z_u \\ P\beta'\sum_0^{\beta l} y^2 \frac{\Delta y}{i} + P\beta\sum_{\beta l}^l yy' \frac{\Delta y}{i} &= Uu_g y_u s = Uy_g u_z s = \phi y_u \end{aligned} \right\} \quad (13)$$

204. *Formation of Force Polygon having $\Sigma\Delta y$ for its Line of Weights.*—Let AB , plate V.a, represent the axial line of a parabolic arch divided into fourteen parts so that their projections upon its chord AB are equal, and which we take as line of weights, whence $\Sigma\Delta y = l$.

This force polygon is required in order to form a cord polygon in connection with another line of multiplying factors y and divisors i , giving intercept between extremes on vertical through the origin A of the y factors, that is Δy and y are the v and a factors of preliminary problem, chap. i., and i the variable pole distance. The force polygon must therefore be formed after the manner of fig. 6 (201).

(The force polygon having m for the mean value of i has not been retained.)

But the sequel will show that it is more convenient to have for pole distance not i but si , giving intercept between extreme rays of cord polygon of

$$\frac{1}{s}\Sigma y \frac{\Delta y}{i}.$$

But

$$s = \Sigma \frac{\Delta y}{i} = \frac{l}{m}.$$

What, then, we require, is a cord polygon, the intercept of whose extreme rays is

$$\frac{m}{l} \Sigma y \cdot \frac{\Delta y}{i} = \frac{\Sigma y \cdot \Delta y}{\frac{l}{m} i} = \frac{\Sigma y \cdot \Delta y}{si},$$

whence, taking as before $\Sigma \Delta y$ for line of weights and increasing the variable i by the ratio $\frac{l}{m}$, we obtain a series of new pole distances, wherefore as

$$m \cdot \frac{l}{m} = l$$

m was the mean pole distance of the force polygon as at first formed, l is the mean pole distance LM , and L the mean pole as now modified.

205. *Formation of First Cord Polygon giving for Ordinate an Expression Proportional to the Left-hand Member of Equation (13).*—Construct a cord polygon (lowermost in figure) corresponding to the force polygon of last article, having

$$\Delta y, 2\Delta y, 3\Delta y, \dots$$

that is the values of y for new line of factors and its sides at right angles to the rays of the force polygon, and bounded by the verticals through the end points of Δy . Then we have the intercepts upon the verticals through A and B taken successively as origin, between any two sides prolonged, equal to

$$\frac{y\Delta y}{si} \text{ and } \frac{y'\Delta y}{si},$$

one of these is marked upon the figure, viz.

$$\frac{y_7\Delta y}{sl_7}.$$

The summation of these intercepts on both verticals through A and B for any division of l into βl and $\beta' l$ is

$$\frac{1}{s} \left\{ \sum_0^{\beta l} y \frac{\Delta y}{i} + \sum_{\beta l}^l y' \frac{\Delta y}{i} \right\}.$$

This summation is marked on the figure for $\overline{5,6}$, whence $\beta = \frac{5}{14}$ and $\beta' = \frac{9}{14}$.

From this, by means of lemma (192), the moment ordinate of the cord polygon is found to be

$$u'_g = \frac{1}{s} \left\{ \beta' \sum_0^{\beta l} y \frac{\Delta y}{i} + \beta \sum_{\beta l}^l y' \frac{\Delta y}{i} \right\} \quad . \quad . \quad . \quad (14_1)$$

This ordinate is called u'_g , for comparing its value in this equation with equation 13, we see that it is proportional to u_g .

206. *Vertical through Centre of Gravity of $\Sigma \frac{\Delta s}{si}$.*—The intersection of the outermost sides of this cord polygon gives a point in the vertical O_y through the centre of gravity of $\Sigma \frac{\Delta s}{si}$, for there

$$\Sigma y \frac{\Delta y}{si} = 0,$$

reckoning the origin in this vertical, whence we have the y_g ordinate, that is, ordinate of G from z .

207. *Horizontal through Centre of Gravity of $\Sigma \frac{\Delta s}{si}$.*—Construct a second cord polygon to the same force polygon having

$$\Delta z, 2\Delta z, 3\Delta z, \dots$$

that is, the values of z for new line of factors, its sides parallel to the rays of the force polygon, and bounded by the horizontals through the end points of Δz (see extreme left of arch in figure). Then we have the intercepts upon the horizontals upon the cord of the bow between any two successive sides prolonged, equal to

$$\frac{z \Delta y}{si}.$$

The intersection of the outermost sides of this cord polygon gives a point O_z in the horizontal through the centre of gravity of $\Sigma \frac{\Delta s}{si}$, for there

$$\Sigma z \frac{\Delta y}{si} = 0,$$

reckoning the origin in this horizontal, whence we have the z_g ordinate, that is, the ordinate of G from y .

The centre of gravity G of $\Sigma \frac{\Delta s}{si}$ is now completely determined.

208. *Formation of Force and Cord Polygon, giving for Ordinate an Expression Proportional to Left-hand Member of Equation (13₃).*—Taking the intercepts

$$\frac{1}{s} \sum y \frac{\Delta y}{i}$$

on the vertical through A for the line of weights of a new force polygon y_g for the pole distance O_{y_g} for pole, and the values of y for the new line of factors, we obtain a cord polygon (second from bottom of figure), whose intercepts between two successive sides on the verticals through A and B are

$$y \frac{y \Delta y}{y_g s i} \text{ and } y' \frac{y \Delta y}{y_g s i}.$$

One of these is marked on figure, viz. for $\overline{6,7}$.

The ordinate of this cord polygon at any point distant βl from A and $\beta' l$ from B after art. 187, is

$$u'_z = \frac{1}{s} \left\{ \beta' \sum_0^{\beta l} y^2 \frac{\Delta y}{y_g i} + \beta \sum_{\beta' l}^l y' y \frac{\Delta y}{y_g i} \right\} \quad \dots \quad (14_3)$$

This ordinate is called u'_z , for, comparing its value in this equation with equation 13₃, we see that it is proportional to u_z .

209. *Vertical through Antipole Z of z Axis.*—The sum of the intercepts upon the vertical through A is

$$\frac{1}{s} \sum y^2 \frac{\Delta y}{y_g i},$$

and the following proportion exists between the force and cord polygons

$$\frac{1}{s} \sum \frac{y \Delta y}{i} : y_g :: \frac{1}{s} \sum y^2 \frac{\Delta y}{y_g i},$$

or omitting common factors

$$\sum y \Delta y : y_g :: \frac{1}{y_g} \sum y^2 \Delta y : \frac{y_g \sum y^2 \Delta y}{y_g \sum y \Delta y},$$

and

$$\frac{y_g \sum y^2 \Delta y}{y_g \sum y \Delta y} = \frac{y_g (y_g + \frac{k^2}{y_g})}{y_g} = y_g + \frac{k^2}{y_g} = y_z.$$

This last is the expression for the ordinate y_z of the antipole Z of axis z , whence the extreme rays of this cord polygon intersect in a point z' in the vertical through the antipole Z of axis z .

210. *Formation of Force and Cord Polygon giving for Ordinate an Expression Proportional to the Left-hand Member of Equation (13₂).*—Taking the intercepts (207),

$$z \cdot \frac{\Delta y}{s i}$$

on the y axis for the line of weights, and z_g for pole distance and the values of y for the new line of factors, construct a cord polygon whose sides are at right angles to the corresponding rays of the force polygon, and whose intercepts upon the verticals through A and B are

$$y \frac{z \Delta y}{z_g s i} \text{ and } y' \frac{z \Delta y}{z_g s i},$$

then the ordinates of the cord polygon at any point distant βl from A is after (2)

$$u'_y = \frac{1}{s} \left\{ \beta' \sum_0^{\beta l} y z \frac{\Delta y}{z_g i} + \beta \sum_{\beta l}^l y' z \frac{\Delta y}{z_g i} \right\} \quad \dots \quad (14_2)$$

This ordinate is called u'_y , for, comparing its value in this equation with equation 13₃, we see that it is proportional to u_y .

211. *Formation of Cord Polygon in order to obtain a Parallel to y Axis going through Antipole Y of Axis y .*—Taking the same force polygon as in last article, and the values

$$\Delta z, 2\Delta z, 3\Delta z \dots$$

of z for the new line of factors, we obtain a cord polygon (immediately left of arch in figure) the sum of whose intercepts on the y axis, is

$$\frac{1}{s} \sum \frac{z^2 \Delta y}{z_g i},$$

one of these for $\bar{3}', 4'$ is marked on figure, and one half of cord polygon is only shown, but being symmetrical it was not necessary

to finish it, and the following proportion exists between force and cord polygons

$$\frac{1}{s} \cdot \Sigma z \cdot \frac{\Delta y}{i} : z_g :: \frac{1}{s} \Sigma z^2 \frac{\Delta y}{z_g i},$$

or omitting common factors

$$\Sigma z \cdot \Delta y : z_g :: \frac{1}{z_g} \Sigma z^2 \Delta y : \frac{z_g \Sigma z^2 \Delta y}{z_g \Sigma z \cdot \Delta y},$$

and

$$\frac{z_g \Sigma z^2 \Delta y}{z_g \Sigma z \Delta y} = \frac{z_g (z_g + \frac{k^2}{z_g})}{z_g} = z_g + \frac{k^2}{z_g} = z_y,$$

the expression for the ordinate z_y of the antipole Y of the axis Y , whence the extreme rays of this cord polygon intersect in a point Y' in the horizontal through the antipole Y of axis y .

212. *Axes of the Central Ellipse.*—By means of pole and antipolar we obtain the two principal axes of the central ellipse, the arcs necessary to their construction are drawn in. The ellipse has more of a demonstrative than of a practical use.

213. *Direction and Value of Force U.* Collecting the Three last Equations, $14_1, 14_2, 14_3$

$$u'_g = \frac{1}{s} \left\{ \beta' \Sigma_0^{\beta i} y \cdot \frac{\Delta y}{i} + \beta \Sigma_{\beta i}^i y' \frac{\Delta y}{i} \right\} \dots (1)$$

$$u'_y = \frac{1}{s} \left\{ \beta' \Sigma_0^{\beta i} y z \frac{\Delta y}{z_g i} + \beta \Sigma_{\beta i}^i y' z \frac{\Delta y}{z_g i} \right\} \dots (2) \dots (14)$$

$$u'_z = \frac{1}{s} \left\{ \beta' \Sigma_0^{\beta i} y^2 \frac{\Delta y}{y_g i} + \beta \Sigma_{\beta i}^i y' y \frac{\Delta y}{y_g i} \right\} \dots (3)$$

Comparing these with equations 13, we can write

$$\begin{aligned} Pu'_g &= Uu_g \\ Pu'_y &= Uu_y \dots \dots \dots (15) \\ Pu'_z &= Uu_z \end{aligned}$$

whence we have

$$\frac{u_g}{u'_g} = \frac{u_y}{u'_y} = \frac{u_z}{u'_z} = \frac{P}{U} \dots \dots \dots (16)$$

and as u_g, u_y, u_z are the perpendiculars upon the direction of U from the three known points G, Y, Z , and this equation (16) shows that u'_g, u'_y, u'_z are proportional to them, we can (fig. 12a, *Proj. Geom.*) construct the U line.

The value of U can be constructed by means of similar triangles from any of the three proportions in (16). It is shown constructed for $P_{4,5}$ by means of the proportion $\frac{u_g}{u'_g}$.

214. *Envelope of the U Lines.*—Constructing a sufficiently large number of these lines, we obtain for their envelope a flat curve concave toward the bow. This curve can be recognized in figure. The $U_{5,6}$ line is drawn in.

215. *Total Reaction \tilde{U} at One of the Springings arising from a Force P .*

First method of constructing.—The reaction U which we have obtained is that part of the whole which resists form alteration. The other part of the reaction is the vertical reaction to P at $A = P\beta'$, and \tilde{U} is the resultant of U and $P\beta'$ and a point in its line of action is necessarily the point (U, z) where U cuts z whence its line of action and the point (P, \tilde{U}) where that line of action cuts the force, P can be obtained.

Second method of constructing.—As

$$\sum_0^{\beta l} = \sum_0^l - \sum_{\beta l}^l$$

let us substitute this equivalent in equations 14, introducing the bending force P on the one side and the resisting force U on the other, we obtain after a simple reduction.¹

¹ The form in equa. 14 is

$$\beta \sum_0^{\beta l} y \cdot dy + \beta \sum_{\beta l}^l y \cdot dy$$

in which for the form $\sum_0^{\beta l}$ we have to substitute $\sum_0^l - \sum_{\beta l}^l$ giving the expression

$$\left. \begin{aligned} \frac{1}{s} \left\{ P\beta' \sum_0^l y \frac{\Delta y}{i} - P \sum_{\beta l}^l (y - \beta l) \frac{\Delta y}{i} \right\} &= Uu_g \\ \frac{1}{z_g s} \left\{ P\beta' \sum_0^l y z \frac{\Delta y}{i} - P \sum_{\beta l}^l (y - \beta l) z \frac{\Delta y}{i} \right\} &= Uu_y \\ \frac{1}{y_g s} \left\{ P\beta' \sum_0^l y^2 \frac{\Delta y}{i} - P \sum_{\beta l}^l (y - \beta l) y \frac{\Delta y}{i} \right\} &= Uu_z \end{aligned} \right\} \quad (17)$$

Now,

$$\left. \begin{aligned} \sum_0^l y \cdot \frac{\Delta y}{i} &= y_g \\ \sum_0^l y z \frac{\Delta y}{i} &= \beta' y_g \sum_0^l z \frac{\Delta y}{i} = y_g z_g s \\ \sum_0^l y^2 \frac{\Delta y}{i} &= y_g y_z s \end{aligned} \right\} \quad (18)$$

whence we may write equations 17

$$\left. \begin{aligned} P\beta' y_g - \frac{1}{s} \left\{ P \sum_{\beta l}^l (y - \beta l) \frac{\Delta y}{i} \right\} &= Uu_g \\ P\beta' y_g - \frac{1}{z_g s} \left\{ P \sum_{\beta l}^l (y - \beta l) z \frac{\Delta y}{i} \right\} &= Uu_y \\ P\beta' y_z - \frac{1}{y_g s} \left\{ P \sum_{\beta l}^l (y - \beta l) y \frac{\Delta y}{i} \right\} &= Uu_z \end{aligned} \right\} \quad (19)$$

$$\begin{aligned} &\beta' \sum_0^l y \cdot dy - \beta' \sum_{\beta l}^l y \cdot dy + \beta \sum_{\beta l}^l y' \cdot dy \\ &= \beta' \sum_0^l y \cdot dy - \beta' \sum_{\beta l}^l y \cdot dy + \beta \sum_{\beta l}^l (l - y) dy \quad (a). \end{aligned}$$

Now

$$\begin{aligned} -\beta' y \cdot dy + \beta(l - y)dy &= -\beta' y \cdot dy + \beta l dy - \beta y \cdot dy \\ &= -y \cdot dy + \beta l dy \\ &\text{as } (\beta + \beta') = 1, \end{aligned}$$

whence expression (a) can be written

$$\beta' \sum_0^l y \cdot dy - \sum_{\beta l}^l (y - \beta l) dy \quad (b)$$

under which form equations 17 are written.

and the second expression in the left-hand member of each of these equations, viz.

$$\left. \begin{aligned} \frac{1}{s} \sum_{\beta l}^l (y - \beta l) \frac{\Delta y}{i} &= u''_g \\ \frac{1}{z_g s} \sum_{\beta l}^l (y - \beta l) z \frac{\Delta y}{i} &= u''_y \\ \frac{1}{y_g s} \sum_{\beta l}^l (y - \beta l) y \frac{\Delta y}{i} &= u''_z \end{aligned} \right\} \dots \dots \dots (20)$$

are evidently the intercepts between the extreme rays of cord polygons having y factors, in regard to βl . These values are shown marked on these figures for $P_{5,6}$, and we may write equations (19) thus

$$\left. \begin{aligned} P\beta'y_g - Pu''_g &= Uu_g \text{ or } P\beta'y_g - Uu_g = Pu''_g \\ P\beta'y_y - Pu''_y &= Uu_y \text{ or } P\beta'y_y - Uu_y = Pu''_y \\ P\beta'y_z - Pu''_z &= Uu_z \text{ or } P\beta'y_z - Uu_z = Pu''_z \end{aligned} \right\} (21)$$

Now y_g, y_y, y_z are the perpendiculars from G, Y, Z upon z , that is, upon the line of vertical reaction $P\beta'$, whence the first terms of the left-hand members of these equations are respectively the moments of force $P\beta'$ around G, Y , and Z . In the same manner, Uu_g, Uu_y, Uu_z are the moments of U around G, Y, Z , and the algebraic sums of these moments are equal to the moments of their respective resultants around the same points. Let \tilde{U} be that resultant $\tilde{u}_g, \tilde{u}_y, \tilde{u}_z$, the perpendiculars from G, Y , and Z upon its direction, then

$$\left. \begin{aligned} \tilde{U}\tilde{u}_g &= Pu''_g \\ \tilde{U}\tilde{u}_y &= Pu''_y \\ \tilde{U}\tilde{u}_z &= Pu''_z \\ u''_y &= u''_g \end{aligned} \right\} \dots \dots \dots (22)$$

whence u''_g, u''_y, u''_z are proportional to the perpendiculars upon \tilde{U} from G, Y , and Z , and the direction of \tilde{U} can be constructed (fig. 12a *Proj. Geom.*). This construction is shown for $P_{5,6}$.

216. *Relation between the u Intercepts.*—The following relation existing between the u intercepts furnishes a means of controlling the accuracy of the cord polygons which determine them.

$$2u_g = u'_z + (u'_z) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (1)$$

$$2u_g = u''_z + (u''_z) \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (2)$$

On account of symmetry we have

$$\sum_0^{\beta l} y^2 \frac{\Delta y}{i} = \sum_{\beta l}^l y' z \frac{\Delta y}{i}$$

and

$$\sum_{\beta l}^l y y' \frac{\Delta y}{i} = \sum_0^{\beta l} y y' \frac{\Delta y}{i},$$

we have also

$$\beta' l = (1 - \beta l)$$

$$y' = l - y$$

$$y + y' = 2y_g$$

whence

$$\begin{aligned} u'_z + (u'_z) &= \frac{1}{sy_g} \left[\beta' \sum_0^{\beta l} y^2 \frac{\Delta y}{i} + \beta \sum_{\beta l}^l y y' \frac{\Delta y}{i} \right] \\ &+ \frac{1}{sy_g} \left[\beta \sum_{\beta l}^l y'^2 \frac{\Delta y}{i} + \beta' \sum_{\beta l}^l y y' \frac{\Delta y}{i} \right] \\ &= \frac{1}{s} \left[2\beta' \sum_0^{\beta l} y \frac{\Delta y}{i} + 2\beta \sum_{\beta l}^l y' \frac{\Delta y}{i} \right] = 2u'_g \cdot \quad (1) \end{aligned}$$

Again, we have from the figure

$$(u''_z) = \frac{1}{sy_g} \sum_0^{\beta l} (\beta' l - y) y \cdot \frac{\Delta y}{i} = \frac{1}{sy_g} \sum_{\beta l}^l (y - \beta l) y' \frac{\Delta y}{i}$$

also

$$u''_z = \frac{1}{sy_g} \sum_{\beta l}^l (y - \beta l) y \frac{\Delta y}{i}$$

adding

$$u''_z + (u''_z) = 2u''_g \cdot \cdot \cdot \cdot \cdot \cdot \cdot \quad (2)$$

217. *Reaction Intersection Curve, Envelope of Force U and Envelopes of the Reactions \tilde{U} and \tilde{U}' .*—The locus of intersection of \tilde{U} and \tilde{U}' is a flat curve over the arch which we may call the *reaction intersection curve*. The envelope of the lines U is a flat elliptical curve, convex downward. The envelopes of the lines \tilde{U} and \tilde{U}' are two branches of a curve with the verticals through A and B for asymptotes.

218. *Interpolating U , \tilde{U} , \tilde{U}' Lines.*—Having now determined the above curves to a sufficient degree of approximation, we can interpolate any values of the U , \tilde{U} , \tilde{U}' lines, by making the three to roll upon their envelopes and intersect in pairs on the reaction intersection curve, and on the verticals through A and B .

219. *Lines of Pressure in the Arch.*—As any corresponding U , \tilde{U} , \tilde{U}' belong only to one P (for it is only the moments of one P which are introduced into equations); now let the axis of the arch be the line of pressures of the structural weight. We can obtain the various U belonging to it, denominating them *U , ${}^*\tilde{U}$, ${}^*\tilde{U}'$; *U acts along AB , ${}^*\tilde{U}$ and ${}^*\tilde{U}'$ act along the tangents to the axis of the bow at A and B , whence for any P compounding the U lines belonging to it with the *U , we obtain the values of $'U$ ($'U$ denominating total value of U) and the points in the verticals of A and B where they intersect, together with a point in the resultant of *P and P . We are now in a position to construct a line of pressures, which must fall within the trace of the core of the cross section.

220. *Forces Generated by Expansion from Increase of Temperature.*—The hypothesis here assumed is that the crown of the arch does not rise, and that consequently the centre of gravity G does not alter its place. The end A slides upon the horizontal outwards. From B towards A upon the axis of the expanded bow lay off the length of the original bow, giving upon that axis a point A' , from A' draw a horizontal, from the original A a vertical, cutting the horizontal from A' in D (see figure 3, page 267), whence we have η and ζ , U also goes through G and ${}_v U = 0$.

From equations (10) we have

$$\frac{u_y \cdot z_g}{u_z \cdot y_g} = \frac{\eta}{\zeta}$$

and substituting the value of s in (193, i.) we have

$$\frac{EI}{\Sigma ds} \cdot \frac{\eta}{u_y z_g} = \frac{EI}{\Sigma ds} \cdot \frac{\zeta}{u_z y_g} = U.$$

Upon two parallels through Y and Z lay off the lengths η and $\frac{\zeta z_g}{y_g}$, the intersection of their uniting line with the line YZ gives a point in U which, on account of $u_g = 0$, unite with G . This is the direction line of U . The value of U can be best obtained by calculation, η and ζ being supposed known. The linear coefficient of the expansion of iron is 0.0000122 per degree centigrade, and the range of temperature may be safely taken at 50° . When these U have been found they can be compounded with the tU .

221. *Arch Hinged at the Springings A and B.*—In this case, all U must go through A and B which must be in the axis of the bow, and u_g, u_y, u_z , are constant for all U .

Hence we can only satisfy the second of the three equations (13) which substituting from equation 14, becomes for all U

$$u_y U = u_y P.$$

As u_y is constant we find the direction lines of \tilde{U} and \tilde{U}' in the simplest manner, by accepting for P the length u_y , whence we have

$$U = u'_y.$$

Lay off u'_y from A horizontally and join to it the vertical $\beta'P = \beta'u_y$, constructing as upon Plate VII. shown upon the third cord polygon from bottom. Their resultant is \tilde{U} in direction, which cuts P upon the reaction intersection curve.

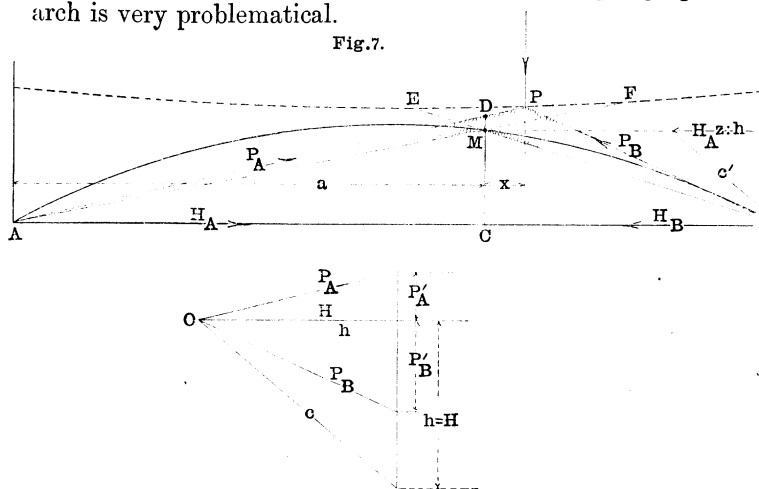
In place of the U envelopes we have the two points A and B , all other constructions are as formerly.

222. *Practical Construction of Arches Hinged at Springing.*—

The true axis of these structures being along the centres of gravity of the cross sections of horizontal member and arch, if there were an independent horizontal member sufficiently great so as to raise the true axis above the axis of the arch, U would not go through A and B . The necessity for a weighty horizontal member is often obviated by making the depth of the upper part of the bow very great. A conspicuous instance of this treatment is found in the bridge over the Douro near Oporto. Another and elegant instance will be found in the bridge over the Ruhr, near Düsseldorf on the Duisburg and Qaackenbrück Railway.¹

The necessity for resorting to hinging the springings of an arch is very problematical.

Fig. 7.



223. *Unfavourable Conditions of Loading.*— AMB is axis of an arch and likewise cord polygon of its structural weight; EF is reaction intersection line, MD is therefore bending moment : h brought upon any point as M (44) ; we see at once that a force at E or F would produce no bending moment upon M , and that exterior to E and F the bending moment changes sign. We have two maximums for the point M , first when from E to F is loaded, second, when all exterior to EF is loaded. In the case of the elastic arch we take the point M on the upper limit of the core and AM , BM are the lines of \tilde{U} and \tilde{U}' .

¹ Heinzerling, *Die Eisernen Bogenbrücken*, 1880.

To face page 284.

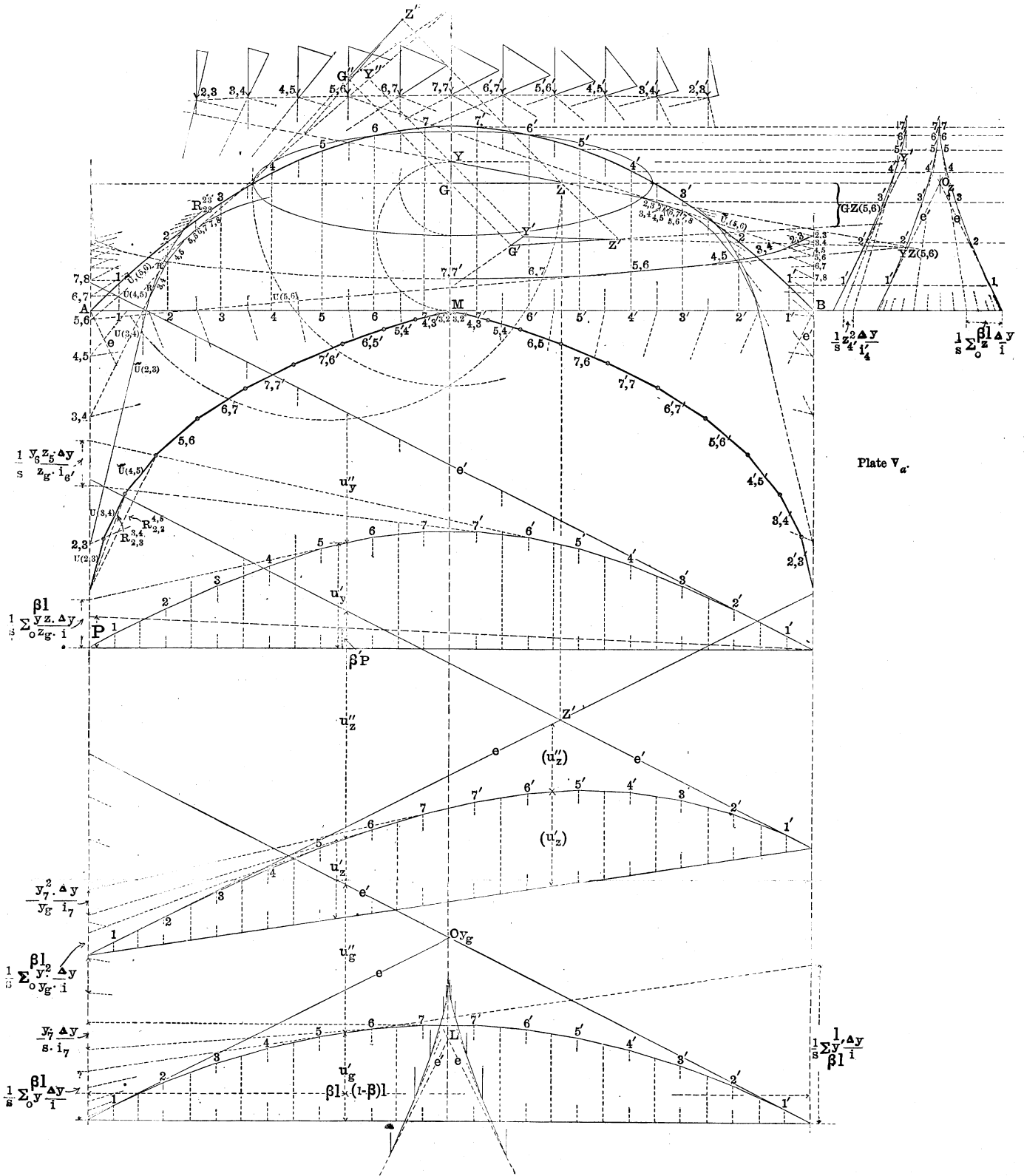
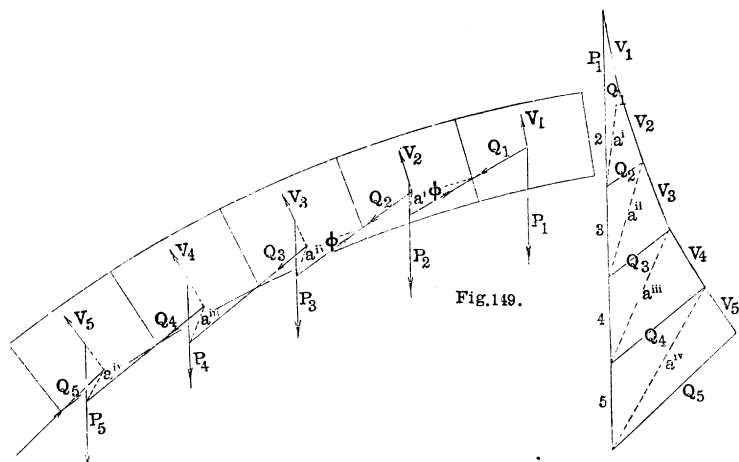


Plate V_a

Section IV.—Arch Centring.

224. *Forces resisted by an Arch Centring.*—Beginning with P_1 (fig. 149) the reaction V_1 upon the centring is normal to it. As the weight P tends to make V slip downwards, it is resisted by friction between the surfaces of the joint acting upwards, wherefore the tangential action is downwards, equal and opposite to the friction, whence the reaction Q of the joint is upward, making an angle ϕ with the normal, and P_1 , V_1 , and Q_1 are in equilibrium.

Q_1 and P_2 give a resultant a' , whose line of action cutting line of action of V_2 gives a point of Q_2 which carry through the joint under V_2 , making an angle ϕ with the joint.

But the direction of Q_2 cuts the intrados, so after having drawn in a'' and obtained a point in Q_3 we no longer draw in Q_3 at an angle ϕ with the next joint V_3/V_4 , but taking for another point of Q_3 the lower end of the joint V_3/V_4 draw it in, and thus proceed.

i. Drawing in Q at an angle ϕ with the joint when Q meets P within the archstone. ii. Drawing in Q through the point (V^{n+1} , a_n) and the lower end of the next joint.

The proceeding requires no formal demonstration.

CHAPTER VI.

THE SUSPENSION BRIDGE AND AUXILIARY GIRDERS.

Section I.—Suspension Bridge.

225. *Curve assumed by Main Chains of Suspension Bridge.*—The lines of the main chains of a suspension bridge are the lines of a cord polygon. If the load, including the structural weight, be equally distributed (46, II. and 48), that polygon becomes a polygon circumscribing a parabola, and for a first approximation this would be the curve chosen.

226. *Method of Obtaining Curve of Main Chains.*—In fig. 150 we have, for the purposes of a diagram, accepted the gross weights, and modified the elevation of an actual suspension bridge. AB is the line of weights of the force polygon, corresponding to the cord polygon formed by the chain of the suspension bridge, this cord polygon, being under any conditions of distribution of its weight, thus obtained. At any point, say (10, 11) on the horizontal line h , obtained by projecting the point (10, 11) on the elevation of the bridge downward, the vertical ordinate to that point is composed of the following

$$\Sigma_{21}^{11}a, + \Sigma_{21}^{11}b, + \Sigma_{21}^{11}c,$$

the accumulated weights, respectively of (*a*) proof weights, (*b*) platform, (*c*) chains, rods, &c., a, b, c their respective increments added to the ordinate at the point (10, 11) by the length (10) of the bridge, so that, by continuing to add the increments for the lengths 9, 8, 7, 6 we at last obtain the ordinate AB of total weight.

If all the weights were uniformly distributed, then the resulting line AC would be accurately a straight line, but if the varying obliquity of the chains, their possibly varying section, the gradually decreasing lengths of the suspension rods be taken into account, the distribution is no longer uniform, the line AC becomes a flat curve and the parts 1, 2, 3, 4 of the line of weights AB are obtained by projecting horizontally the points (1, 2) (2, 3) (3, 4) of the curve AC horizontally upon the line AB , as shown for the points (8, 9) (9, 10).

The curve of the chains of the suspension bridge being a cord polygon, we have its line of weights AB given, a point V in one of its extreme rays e , and the direction e' , and extreme point H of the other, whence we can obtain the pole O , and with it draw in the cord polygon or curve of chains.

This can easily be done by taking a provisional pole O' (preferentially in the horizontal AO , so that its extreme ray may be also He'). Let its other extreme ray $V'(e)$ cut He' in R . The extreme ray Ve of the chain curve cuts also in R . Draw BO parallel to VR . O is the pole required. From O construct the chain curve. The rays from O measure the tensions in the various links of the chain (22).

227. *Anchoring the Chain.*

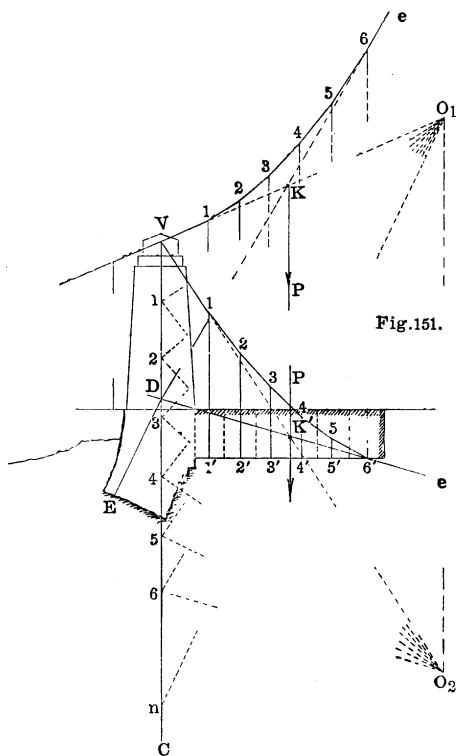
i. *With Vertical Suspension Rods, fig. 151.*—This method is not employed in practice, but introduced as a stepping-stone to the next method.

Let the weight of the masonry behind the piers, to which the chains are anchored, be, for security, so apportioned as to be able to resist twice the tension of the chain at the vertex V of the pier.

For convenience, instead of continuing the line of weights AB (fig. 150) downwards below B , we will make the vertical VC , the line of weights, and VO_1 parallel to BO , and measuring double the tension of BO ; VO_1 is however drawn to a smaller scale. The double tension is approximately 1940 tons.

Let there be, behind the pier, approximately masonry, having a full breadth of 30 feet, giving a breadth of 15 feet for each series of chains. Let this masonry weigh one ton for each foot of depth and length. Let the depth of masonry be 24 feet, and let it be divided into imaginary laminæ, 15 feet long, 15 feet broad, and 24 feet deep, each lamina will weigh 360 tons; then the following construction proceeds upon the supposition that the influence of the weight of the masonry is brought to bear upon the chain.

Take 360 tons, within the points of the compasses, and with them lay off from V upon VC the points 1, 2, 3, . . . and then with the pole O_1 draw the cord polygon, having its angular points in the verticals, drawn through the centres of gravity of



the masonry laminæ, till we arrive at the vertical 6, through the sixth layer, then the continuation of the extreme cord polygon side 6e downward is the line in which the tension is now deflected, and it cuts the line of weights going through the centre of gravity of the pier in the point *D*.

As this is the last deflected line of tension arising from the masonry behind the pier, whose direction continued is within its base, it follows that the masonry must be continued so far back in order to give the required stability.

The line of tension is still further deflected, by means of the masonry in the pier, the measure of whose influence is obtained by drawing from *D*, *DE* parallel to O_1n , giving the modulus of stability in the base of the pier.

In order to bring the weight of the masonry to bear upon the chain, let the chain be anchored at 6' in the masonry, so that *V*

and $6'$ become its two points of support, and attaching, say to the ends of its links, vertical suspension rods $5', 4', 3', 2', 1'$, whose lower ends are fixed in the bases of the successive masonry laminæ.

The chain $V6'e$ is simply another cord polygon similar to $V6e$. These two cord polygons have one point V in common and the other corresponding points $1, 2, 3 \dots$ are in the same verticals. Whence the extreme rays of both cut in the same point D , and as the line of weights and the forces $1, 2, 3 \dots$ are the same in both, and in the same verticals, their resultants P are the same, and the intersections K and K' of the extreme rays in both are in the same vertical, whence the extreme ray of the second cord polygon goes through K' .

The extreme rays of the cord polygon are now given, and its force polygon is now easily found, thus: continue VK' till it cuts the vertical through O_1 in O_2 . This is its pole. The line of weights is the same. The links of the suspension chain $V6'$ being sides of the new cord polygon, can now be drawn in.

ii. *With Oblique Suspension Rods* (*fig. 150.*)—The suspension rods now sustain some portion of the horizontal tension. This however does not alter the extreme rays of the cord polygon nor the pole O_2 . In this case the lengths of the suspension links $\overline{V2}, \overline{2, 3}, \overline{3, 4} \dots$ are arbitrary and given. (The two first masonry laminæ, 1 and 2, are treated together to avoid confusion in the figure.)

Mark off the first link $V2$ of the chain along the exterior ray VK' . Join $2(1, 2)$ for the suspension rod *ii*. From V draw $V2''$ parallel to this suspension rod, then O_22'' gives a parallel to the next link $\overline{2, 3}$, which draw, marking off its length. Join $3, 3'$ for the line of the second suspension rod *iii*, and draw $2''3''$ parallel thereto. Then O_23'' is a parallel to the link $\overline{3, 4}$, which draw, marking off its length.

Proceeding in this manner to the end, we obtain a force polygon $C_2(V, 2'', 3'' \dots 5'')$ whose rays measure the tensions in the links of the suspension chain. The sides $\overline{V2''}, \overline{2''3''}, \overline{3''4''} \dots$ measure the tensions in the suspension rods.

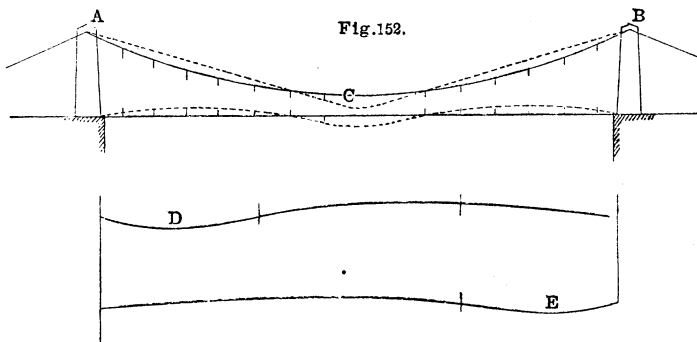
228. *Suspension Bridge with Oblique Suspending Rods.*—A similar transformation may be carried out with regard to the chains and suspension rods of the bridge itself, giving the well known variety called “Dredges Suspension Bridge.”

229. *Anchoring by means of Vaulted Masonry behind the Pier.*—In this case the tension of the anchored suspension chain takes the place of the reaction of the abutment of the arch, which must be so designed as to give a line of resistance falling within the central third of the arch ring and within the central third of the base of the pier.

230. *Anchoring in Rock.*—This requires no elucidation.

Section II.—Auxiliary Girders of Suspension Bridge.

231. *Reasons why Suspension Bridges require to be Stiffened.*—As the cord or link polygon varies the position of its links according to the position, direction, and value of the pressures to which it is subjected, so the chains of a suspension bridge, being such a polygon, alter their position with every partial load brought upon them, and with every different position which that load comes to occupy, and the platform of the bridge being attached to the chains by the suspension rods follows the movements of the chains.



Should a partial load (fig. 152) successively occupy the joints C, D, E , then the chains are successively depressed at these points, and they being of invariable length between the points of support A and B , are consequently raised and straightened on either side of the depression, the platform following the displacement of the chains.

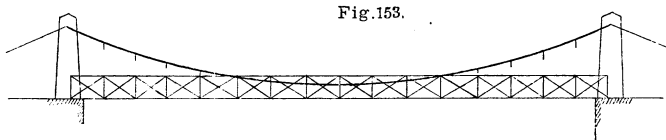


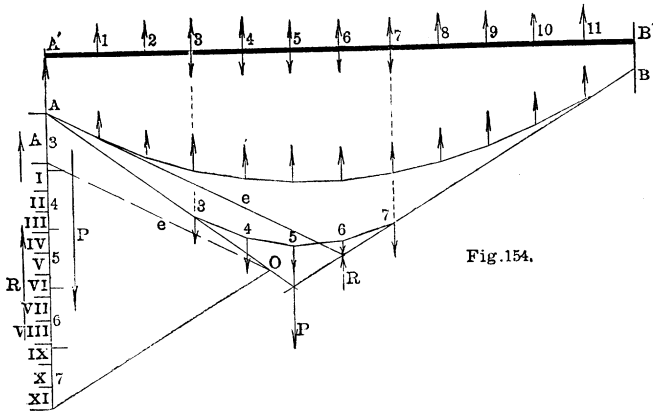
Fig. 153.

232. *Auxiliary Girders employed to Stiffen Suspension Bridge* (fig. 153).—Suspension bridges are now invariably stiffened by longitudinal girders whose duty it is to bear the travelling load.

233. *Points of Support of Auxiliary Girder*.—The points of support of the auxiliary girder are the abutment nearest the centre of gravity of the load, and the suspension rods connected with the points of the girder. The separate value of the reaction at each of the points of suspension depends on the normal form of the chain.

234. *Reactions at Points of Support of an Auxiliary Girder and Cord Polygon of Distributed Pressures*.—If the normal form of the chain is a parabola, then the reactions at the points of suspension are equal, for, the chain being supposed to be held rigid by the girder, the resultant of its reactions must pass through the intersection of the extreme rays of the chain, the chain being in this case the cord polygon of equally distributed pressures.

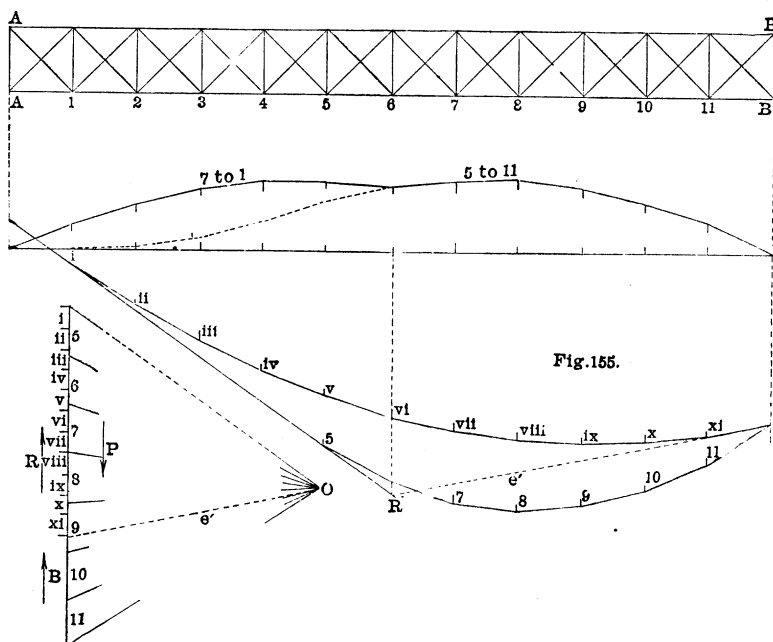
In fig. 154, we have a travelling load P supposed concentrated on the joints 3, 4, 5, 6, 7 of the girder, the points of support of this travelling load are the abutment A nearest P , and the points 1, 2, 3, . . . 11 of the chain, transmitted by means of the suspension rods.



235. *Construction of Reactions of Auxiliary Girder at Points of Support. Cord Polygon of Chain Reactions, and Cord Polygon of Partial Load.*—The construction necessary to find the distributed reaction of this load over the chain joints 1, 2, 3, . . . 11, and over the abutment *A*, are carried out in fig. 154.

The force polygon for the pressure P , i.e. for 3, 4, 5, 6, 7, is first formed, from which we obtain the cord polygon $A, 3, 4, 5, 6, 7, B$; its extreme rays necessarily meeting in the line of action of the resultant P , but the extreme rays of the reactions of the chain joints must meet in the vertical of R , R being known. Then, to the point where the vertical R cuts the extreme ray BP of the travelling weight cord polygon, draw AR . Then the extreme rays of the chain reaction cord polygon are AR, BR , and drawing $O(A, I)$ in the force polygon parallel to AR , then A on the line of weights is the abutment reaction, and the remaining portion of 3, 4, 5, 6, 7 on the line of weights is the chain reaction. Dividing this remaining portion into eleven equal parts, we have the reaction distributed over the joints of the chain, and we are now able to complete the cord polygon of chain reaction.

236. *Measure of Bending Moments of Auxiliary Girder.*—The intercept between the cord polygon of travelling load and the cord polygon of chain reaction at any point, measures the bending moment of the auxiliary girder at that point : pole distance.



237. *Most Unfavourable Disposition of Travelling Load.*—The above construction has been carried out on fig. 155 for the most unfavourable disposition of travelling load coming on to the bridge from *B*, when it extends to joint 5.

That this is the most unfavourable disposition of the travelling load might have been shown by forming a series of similar constructions and then superimposing them.

238. *Measure of Shearing Force in Front of Travelling Load.*—Maximum shearing force occurs at the extremity of the travelling load and must be equal to the sum of the reactions beyond it: thus in fig. 155, the shearing force is the sum of the reactions beyond 5, i.e. $i + ii + iii + iv$.

239. *Auxiliary Shrouds of MR. ORDISH.*—Additional contrivances are often superadded to the auxiliary girders, still further to insure a rigid platform, and that of shrouds radiating from the pier and attached at the other extremities to the stiffening girders, are perhaps at once the most effective and elegant.

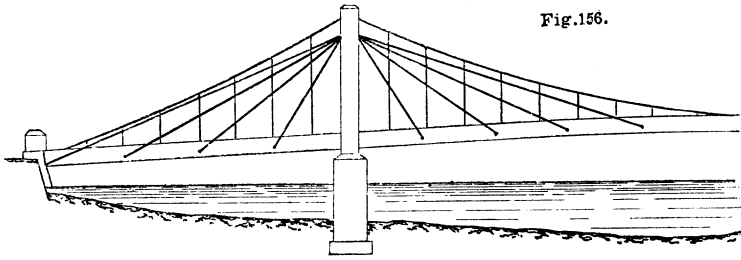


Fig. 156.

We give a skeleton semi-elevation of the Albert Bridge over the Thames at Chelsea, having these auxiliary shrouds.

It is evident that if, at first, a shroud is perfectly straight, the lowering of any point as *A* of the platform cannot be effected, and in order that these shrouds may retain their rectilinearity, they are connected to the suspension rods by a suitable contrivance at every point where a shroud crossed a suspension rod.

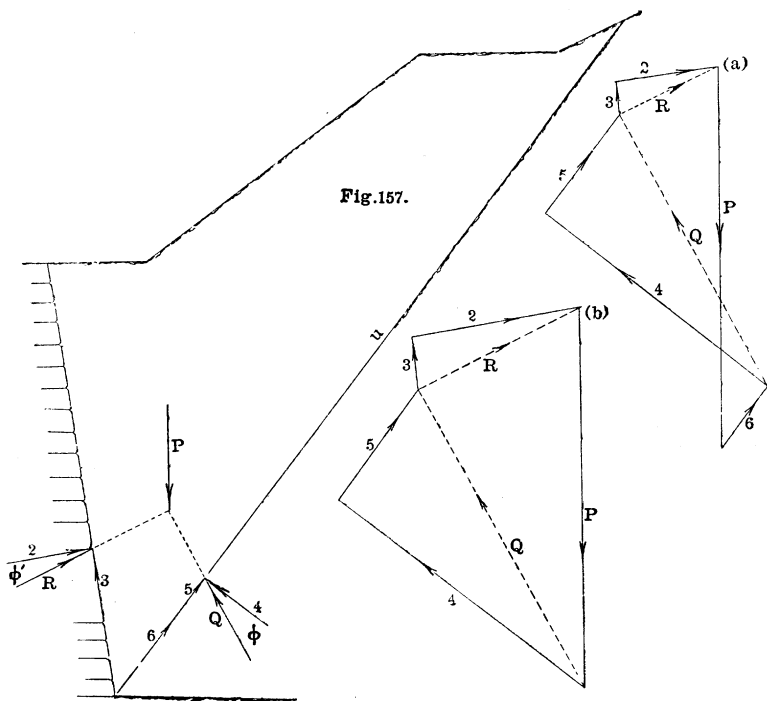
The auxiliary girders are likewise placed a little off the vertical.

The light appearance of the main chain in this bridge brings the shrouds into too strong relief, and mars its otherwise elegant appearance.

240. *Horizontal Main Chains* are sometimes added on either side, to guard against the effects of wind, but their consideration involves no additional methods of procedure.

CHAPTER VII.

ON RETAINING WALLS.

*Section I.—Statement of the Problem.**(a) Active Pressure of Earth upon a Retaining Wall.*

241. *Plane of Rupture.*—Fig. 157 is the cross section of a prism of earth, which will throughout this chapter be supposed one unit in length, and on which are represented the forces called into exercise by its equilibration, when sustained on the one hand by a retaining wall upon which it thrusts, which thrust we shall call its active pressure, and on the other by an arbi-

trarily chosen plane of rupture; for we will, as is generally done, suppose the surface of rupture to be a plane.

242. *The Forces in Equilibrium are (fig. 157) :*

1. The weight P of the prism of earth, and whatever it is loaded with.
2. The normal reaction of the retaining wall.
3. The friction of the prism on the inner surface of the retaining wall, acting upward.
4. The normal reaction of the plane of rupture.
5. The friction of the prism on the plane of rupture, acting upward.
6. The cohesion of the earth on the plane of rupture, acting upward.

Fig. 157*a* represents the force polygon of all these forces. Fig. 157*b* represents the force polygon of the first five of these forces, the sixth or cohesive force being neglected.

In both these force polygons the lines Q and R represent the resultants of 4 and 5, and of 2 and 3 respectively, which we shall call the oblique reactions of the plane of rupture and of the retaining wall, or simply the "reactions." When the force 2 or 4 are spoken of they will always be called the normal reactions.

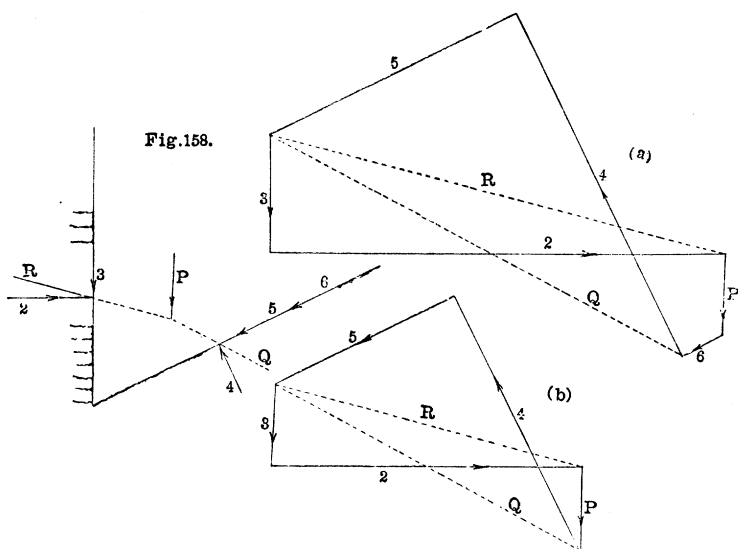
These reactions Q and R make (176) angles with their normal reactions equal the angles of repose, these angles being represented by ϕ and ϕ' ; ϕ being the angle of repose of the prism on the plane of rupture, ϕ' of the prism on the retaining wall. The usual value of ϕ is 35° and of ϕ' 15° .

The force 6 of cohesion for any plane of rupture shall be called K .

The force polygon will, in the following, be considered under the form of the three forces P , Q , R , or of four P , Q , R , K .

(b) *Passive Resistance of Earth.*

243. Fig. 158 is the cross section of a prism of earth sustaining a thrust, which we shall call its passive resistance, such as might arise from the thrust of an arch.



The forces are the same in kind as in thrust, but the forces of friction and of cohesion now act downwards. Figs. 158a and 158b represent the force polygons with, and neglecting cohesion, respectively.

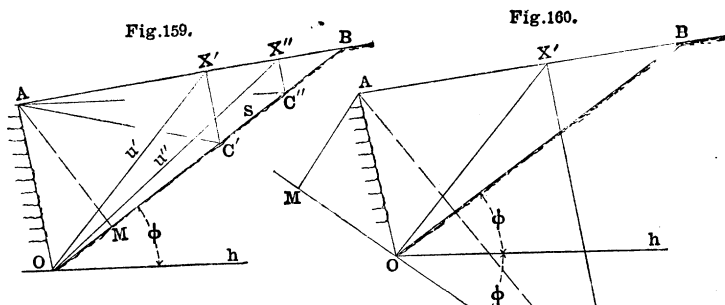
244. *Plane of Rupture of Maximum Thrust.*

(a) *Active Pressure.*—A certain plane of rupture makes reaction R of retaining wall a maximum.

In the foregoing figures the plane of rupture has been drawn in arbitrarily, and R is the reaction of the wall due to that plane of rupture. For as P varies with the plane of rupture and R varies with P , and, as P increases as the inclination of the plane of rupture u decreases, that condition operating alone would increase R , but as P is thus increased, the reaction Q of the plane of rupture u increases and approaches more nearly to P in direction. This operating alone would decrease R . But these two causes operating together determine a certain inclination of u up to which the increase to P adds more to R than the increase and change in direction of Q take from it, but beyond which the influence of P in increasing is less than the influence of Q in

diminishing, till the plane of rupture coinciding with the plane of repose, beyond which the influence of an increase of P is zero.

(b) *Passive Resistance*.—A similar statement may be made regarding the plane of rupture in sustaining a thrust.



245. *Lines P representing Weight of Earth Prism for any Planes of Rupture.*
— OB or s (fig. 159) is the plane of repose, AO inner surface of retaining wall, AB terrain line. Letting fall a perpendicular upon OB , the value of the triangle AOB is represented by

$$\frac{AM}{2} \cdot OB$$

and let ϵ be the weight per cubic unit of earth, then

$$P = \epsilon \cdot \frac{AM}{2} \cdot OB.$$

Let OX' , OX'' be any planes of rupture necessarily steeper than the plane of repose, then drawing $X'C'$, $X''C''$. . . parallel to AO

$$\text{triangle } AOX' = \text{triangle } AOC'$$

we have now a series of triangles AOB , AOC' , AOC'' having the same altitude AM , which are to one another as their bases, or

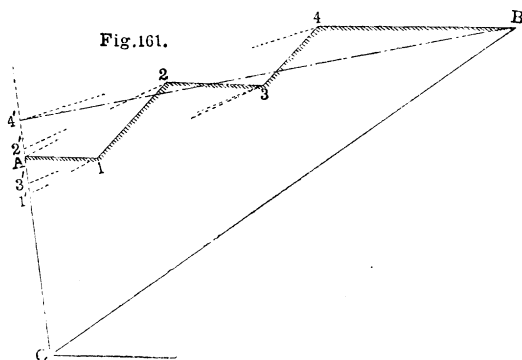
$$OB : OC' : OC'' : \dots :: P : P' : P'' : \dots$$

so that OB , OC' , OC'' represent to the base $\frac{AM}{2}$ the areas

$\frac{P}{\epsilon}$ of all the triangular prisms AOB , AOC' , AOC''

Fig. 159 represents the construction for a prism thrusting on the wall AO .

Fig. 160 represents a similar construction for a prism sustaining a thrust where the values for $\frac{P}{\epsilon}$ are found upon the reflection of the plane of repose, the horizontal h being the reflecting plane, but to which the above words equally apply.



246. *Reduction of an Earth Prism with Irregular Terrain Line to a Triangular.*—This well-known problem requires no demonstration here, but in order to give facility in its application we exhibit it here with a methodical arrangement of symbols.

To reduce the area $OA, 1, 2, 3, 4, \dots B$ (fig. 161) to a triangle, retaining the side OB and the line OA . Number the angles $A, 1, 2, 3, 4 \dots B$, then placing the parallel rulers upon $A2$ draw $1, 1'$ parallel thereto, *i.e.*

Parallel to $A2$ draw $1, 1'$

„ „ $3, 1'$ „ $2, 2'$

„ „ $4, 2'$ „ $3, 3'$

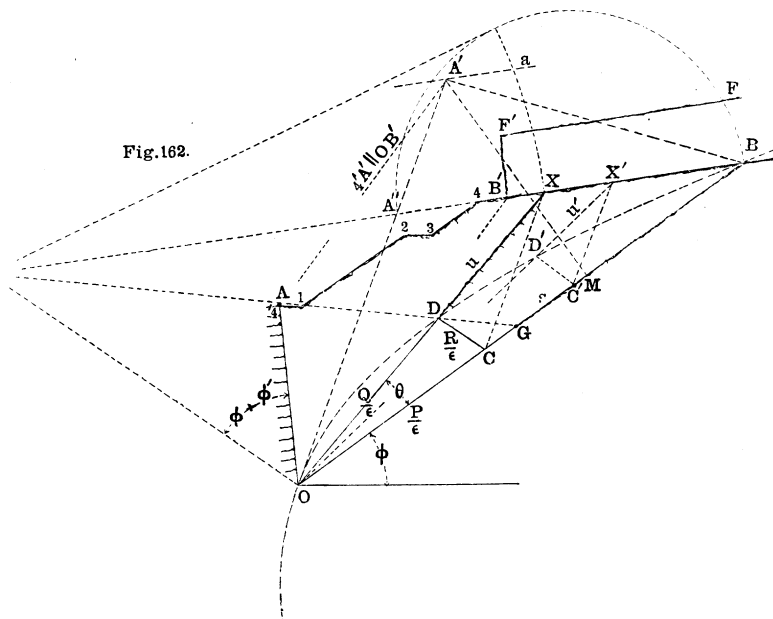
„ „ $B3'$ „ $4, 4'$

Join B and $4'$ by a straight

$O4'B$ is the triangle required.

247. *Transformation of a Loaded Earth Prism to an Appropriate Triangle.*—Fig. 162 represents the section of an earth

Fig. 162.



prism, $OA, 1, 2, 3, 4, B'B$, loaded uniformly from B' , and having this uniform loading represented by an equivalent parallelogram of earth $B'F$. Reduce the many-sided section, $OA, 1, 2, 3, 4, B'$, to an equivalent triangle, $O4'B'$. Draw a line a parallel to $B'B$ or $F'F$, and at the distance $2BF$ from BB' . Draw $4'A'$ parallel to OB' cutting a in A' . A' is the vertex of the triangle required.

For

Triangle $A'BB' = \text{parallelogram } B'F'FB$

being upon the same base and of double altitude; again

Triangle $A'OB' = \text{triangle } O4'B'$

being upon the same base OB' and between the same parallels OB' and $4'A'$ ($4'$ here coincides accidentally with A), whence

$$\begin{aligned} A'BB' &= BFF'B' \\ + A'OB' &= + 4'B'O \\ + BOB' &= + BOB' \end{aligned}$$

or summing

$$A'OB = BFF'B'4'OB = A1234B'F'FBOA.$$

Representative line OC of a loaded prism can now be drawn after the manner of art. 244. For drawing XC , $X'C'$ parallel to $A'O$, we have by similar reasoning as formerly

$$OB : OC' : OC'' \dots :: P : P' : P'' \dots$$

248. *Completing the Force Polygons upon the Lines OC . OC'* *fig. 162.*—Having now the means of projecting the representative value of any P upon the plane of repose in the case of action, and upon its reflection in the case of passive pressure, it is necessary to our future constructions that we complete the force polygon there. As the true position of P is vertical, this is equivalent to turning round the force polygon through an angle $90^\circ - \phi$ in the first case, and $90^\circ + \phi$ in the second.

Let θ be the angle which the plane of rupture makes with the plane of repose, then it can easily be shown from elementary considerations that, in the case of active pressure, the angle which the reaction Q makes with P is also θ in the case of passive pressure $2\phi \pm \theta$.

This can be deduced most easily by the student from figs. 163 and 164, where the values of both given and deduced angles tending finally to this conclusion are marked.

The remaining side R of the force polygon forms an angle (counting turning in the direction of the hands of a watch, as positive) in the case of active pressure $-(90^\circ + \phi')$ with the inner surface OA of the revetment, and requires to be turned round through $90^\circ - \phi$, whence the angle it will form with OA after the turning will be the difference between the angle it makes and the angle it has turned through, that is, as we have given the proper signs to their values, their algebraical sum

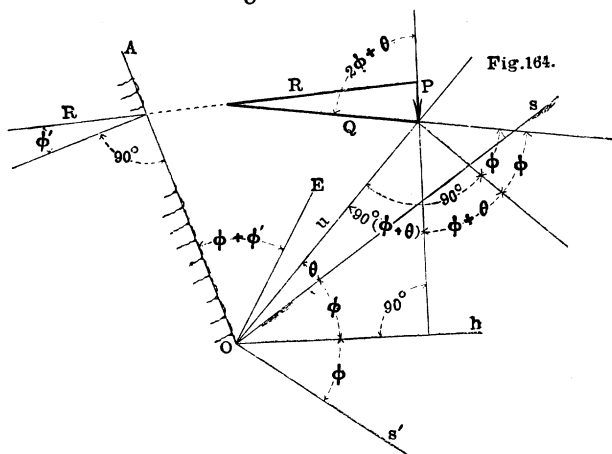
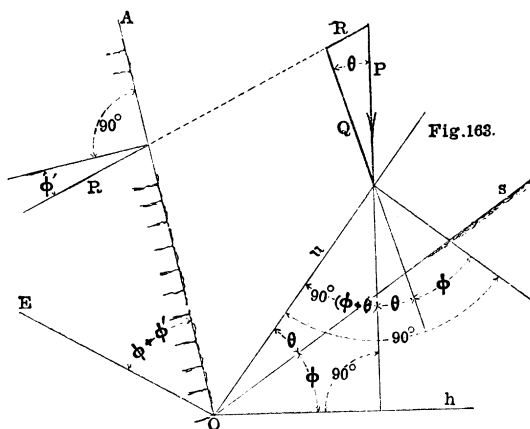
$$-(90^\circ + \phi') + (90^\circ - \phi) = -90^\circ - \phi' + 90^\circ - \phi = -(\phi' + \phi)$$

or it will then make with OA an angle $AOE = -(\phi' + \phi)$ the minus sign showing that OE is to the left of OA .

In the case of passive pressure, the angle which R makes with OA is $-(90^\circ - \phi')$, and it requires to be turned round through an angle $90^\circ + \phi$, whence by similar reasoning it will after turning make with OA an angle AOE

$$= -(90^\circ - \phi') + 90^\circ + \phi = -90^\circ + \phi' + 90^\circ + \phi = \phi' + \phi$$

the plus sign showing that OE is to the right of OA .



OC representing *P* lies on the plane of repose, fig. 162 (or its reflection). From *C* draw *CD* parallel to *OE* cutting the plane of rupture in *D*, *CD* represents *R*, *OD* on the plane of rupture represents *Q*. We can now construct the force polygon for any arbitrary plane of rupture *OX*.

249. *The Locus of the Point D for all Positions of the Plane of Rupture is a Hyperbola.*—If we construct the force polygons for a number of planes of rupture OX, OX' , we obtain a number of points D which lie on a hyperbola ODB going through the points O and B and to which OA' is a tangent.

For the generation of this curve (fig. 165), we have three pencils of rays, one pencil $O(X, X' \dots B \dots)$ one pencil $\tilde{P}(X, X' \dots)$ where \tilde{P} is at infinity, one pencil $\tilde{N}(CD, C'D' \dots)$ where \tilde{N} is at infinity, and intersecting the rays $\tilde{P}X, \tilde{P}X' \dots$ in the points $C, C' \dots$ on OB . It falls therefore under Maclaurin's theorem (*Proj. Geom.*).

For more easy comprehension we have given the homologous circle of this curve, which consider, and enunciate theorem referred to with the letters in our figure. If around a fixed point P we cause a transversal to turn encountering two fixed straights BE, BO in pairs of points C and X and from two fixed points N and O we lead pairs of straights NC, OX the points of intersection D of these pairs of straights lie upon a conic.

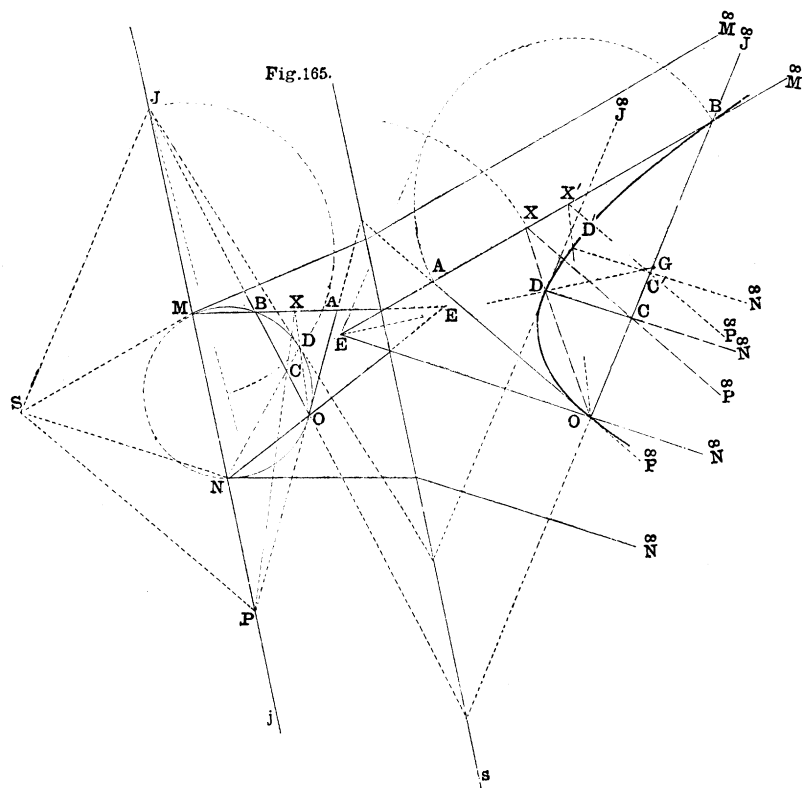
In the case of the figure itself \tilde{N} is at infinity, and as in turning the ray OX round O it becomes parallel to AB , there is a second point \tilde{M} at infinity, it is consequently a hyperbola whose asymptotes meet the curve at infinity in the points \tilde{M} and \tilde{N} . In generating the curve we readily perceive that AO is a tangent meeting the line $\tilde{M}\tilde{N}$ at infinity in the point \tilde{P} , \tilde{P} is therefore at infinity.

250. *Construction of the Maximum Value of CD or R .*—In order to obtain our purpose, which is to find the point D where CD is a maximum, we must construct a tangent $D\tilde{J}$ to the curve parallel to OB whose point of contact is the point D required.

Suppose this done, we have a pencil of three rays in involution $D\tilde{J}, O\tilde{J}$, and the infinitely distant straight $\tilde{M}\tilde{N}$.

This will be more clearly seen from the homologous figure where J is the centre of the pencil of four rays $J(MN, BO, D, D)$ for D is a double point and MN, BO , and DD , are three pairs of points in involution $OM, OE; OB, OA; OX, OX$; are three pairs of conjugate rays. In the figure itself the point \tilde{M} , being at infinity and E its corresponding point, A and B two other corresponding points, X a double point

$$EA \cdot EB = EX^2,$$



whence by the well-known construction (*Euc. iii. 36*), shown in the figure, the point X is found.

Now, drawing the rays OX, XC, CD , we obtain the required point D and the maximum value of CD .

251. *Variations in Construction of Maximum Earth Pressure.*—Projection of the involution from the terrain line to the line of repose and the line EO .

The method above unfolded of constructing the earth pressure by an involution along the terrain line, fig. 166, is the simplest, yet as an involution is projective we can vary our construction in different ways. Of these projections two are found most useful, viz. that along the line EO , and that along the line of repose. The latter may be used advantageously when the point E is at an inconvenient distance.

252. *Lemma.*—The line $\overset{\infty}{J}XL$, fig. 166, necessarily parallel to OB , is the Pascal line of the inscribed hexagon $\overset{\infty}{M}, B, D, D, O, \overset{\infty}{N}$.

Such a hexagon is given in fig. 167, with its homologous circle to assist the conception of the student; bearing in mind that the infinitely small side DD of the hexagon is replaced by its tangent JD . By construction, MN and DD meet in J or the 6th and 3rd sides of the hexagon; by construction also MB and OD meet in X or the 1st and 4th sides of the hexagon, whence by Pascal's theorem the 2nd and 5th, that is BD and ON , meet in L on the same straight with J and X , which therefore must be true of its projection where $\overset{\infty}{J}$ is at infinity.

253. *To Express R in Terms of the Involution on OE (fig. 166).*
—As

$$\frac{P}{\epsilon} = \frac{A'M}{2} \cdot OB \dots \dots \dots (1)$$

so is

$$\frac{R}{\epsilon} = \frac{A'M}{2} \cdot CD \dots \dots \dots (2)$$

and from the foregoing lemma, we have

$$CD : OL :: BC : BO$$

OA' and CX being parallel

$$BC : BO :: BX : BA',$$

$XL, A'N, OB$ being parallel, we have

$$BX : BA' :: OL : ON,$$

wherefore

$$CD : OL :: OL : ON,$$

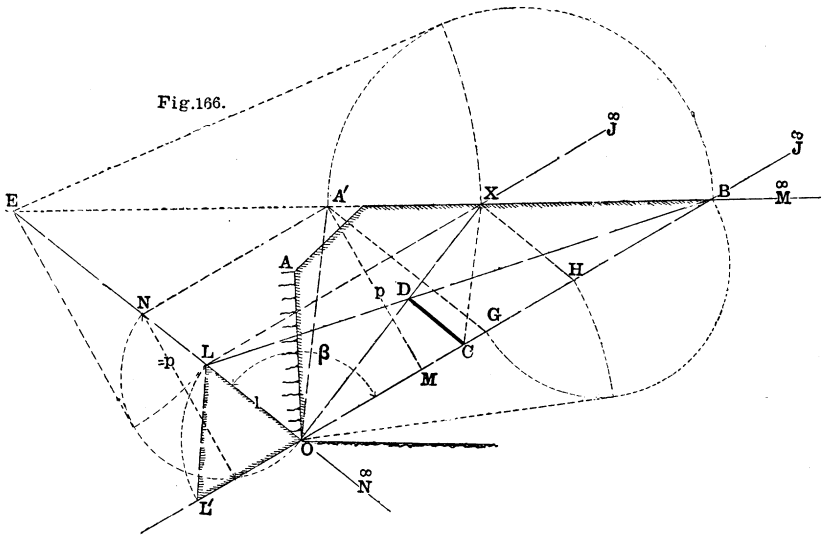
or

$$CD = \frac{OL^2}{ON},$$

substituting this value of CD in equation

$$\frac{R}{\epsilon} = \frac{1}{2} \cdot \frac{A'M}{ON} \cdot OL^2 \dots \dots \dots (3)$$

Fig.166.



but

$$\frac{A'M}{ON} = \sin \beta$$

wherefore

$$\frac{R}{\epsilon} = \frac{1}{2} \sin \beta \cdot OL^2 = \frac{1}{2} \sin \beta \cdot l^2 \quad . \quad . \quad . \quad (4)$$

On OB extended make $OL' = OL$, and join LL' . Whence

$$\frac{R}{\epsilon} = \text{area of triangle } OLL' . \quad . \quad . \quad . \quad (5)$$

OLL' is called the triangle of earth pressure just as AOX is the triangle of earth which presses.

254. *Pressure of Earth per Unity of Area at any Point O in a Retaining Wall (fig. 168.)*—As far as we have gone, for every point O of the wall we have a different base AM , and it is required to find the pressure of area per unity of area at any point independently of any base.

The pressure per unity of area evidently increases directly as the depth, so that that pressure is twice as great at any point O as at half that depth, viz. at

$$\frac{OA}{2}$$

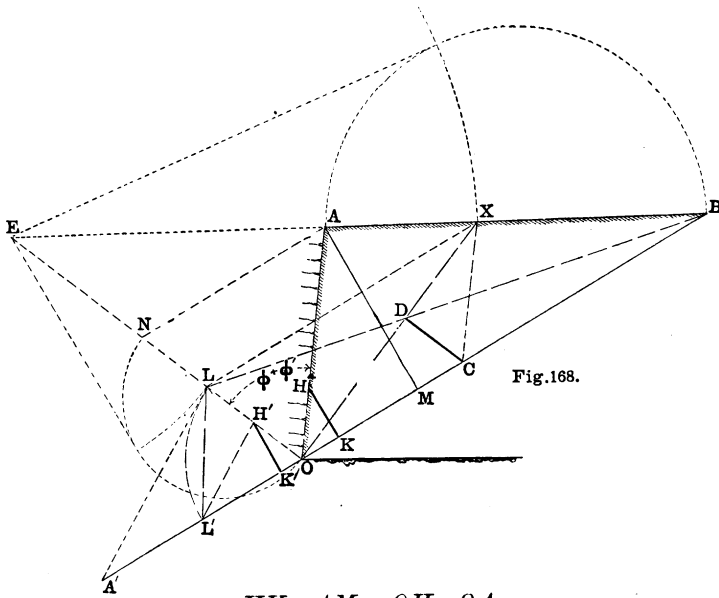


Fig. 168.

$$HK : AM :: OH : OA \\ :: CD : OA,$$

$$\text{or,} \quad HK = \frac{AM \cdot CD}{OA} \quad \dots \quad (7)$$

As this \times by ϵ is the pressure per unity of area immediately at O , and as the rate of pressure increases uniformly, it is the mean rate for an equal portion of wall immediately above and below O , and if for a small portion O may likewise be regarded as a point in the resultant of the pressure distributed over that portion.

Another method of obtaining HK is useful when the point B is inaccessible. On OL' extended, lay off, $OA' = OA$, draw parallels $A'L$ and $L'H'$, then

$$OA' : OL :: OL' : \frac{OL \cdot OL'}{OA'} = \frac{OL^2}{OA} = OH',$$

and product of extremes = product of means,

$$OL^2 = OA' \cdot OH' = OA \cdot OH.$$

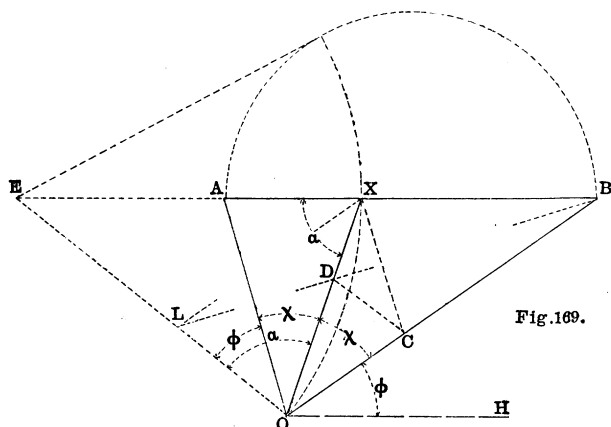
But

$$\frac{R}{\epsilon} = \frac{1}{2} OL^2 \cdot \sin \beta = \frac{1}{2} OA \cdot OH \cdot \sin \beta = \frac{1}{2} OA \cdot H'K',$$

and also $= \frac{1}{2} AM \cdot CD$ (213)

whence $\frac{1}{2} OA \cdot H'K' = \frac{1}{2} AM \cdot CD \dots$

$$HK' = \frac{AM \cdot CD}{OA} = HK \dots$$



255. *Conditions Simplifying Construction. Terrain Line Horizontal and Angle $\phi' = 0$ (fig. 169).*—In this case

$$\angle AOE = \phi; \angle EBO = \angle BOH = \phi,$$

being alternate angles. Consider the two triangles EOB and EOA , $\angle B$ of the first $= \angle O$ of the second, $\angle E$ common to both, they are therefore similar triangles, whence

$$EA : EO :: EO : EB$$

$$EO^2 = EA \cdot EB.$$

But

$$EX^2 \text{ is likewise } = EA \cdot EB;$$

whence

$$EO = EX,$$

and

$$\angle EOX = \angle EXO = \alpha,$$

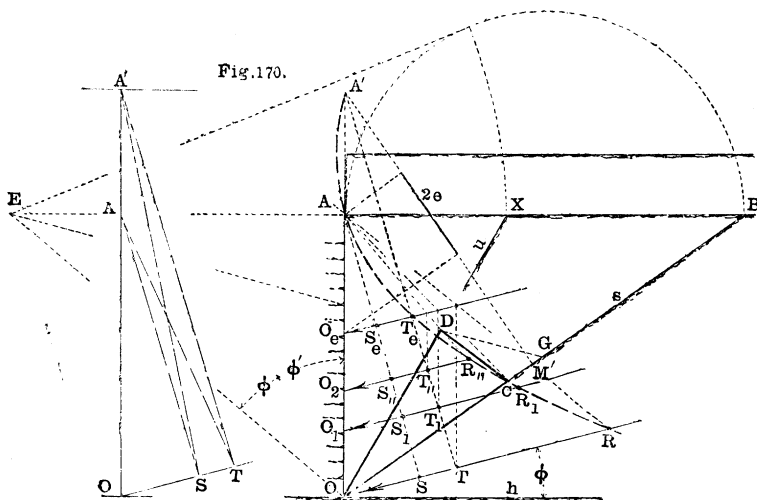
$$\angle AOX = \alpha - \phi,$$

$$\angle OXB = 180^\circ - \alpha,$$

$$\angle XOB = 180^\circ - ((180^\circ - \alpha) + \phi) = + \alpha - \phi,$$

$$\therefore \angle AOX = \angle XOB;$$

wherefore, with the conditions above, the plane of rupture of maximum active pressure bisects the angle formed by inner line of retaining wall and line of repose.



256. *To Reduce a Series of Pressures, CD, for a Series of Points, O, of a Wall with Inner Surface in One Plane, with a Variable Base, AM, to One Base e (fig. 170).*—The following method may sometimes be useful, and supplement art. 245.

Find the value of CD for any convenient point O . Through AO draw $O_1O_2O_3 \dots$ in the direction in which the earth pressure acts. Lay off $OS = CD$, and through A draw AS ; AS will cut off the parallel lines through $O_1O_2O_3 \dots$. OS , O_1S_1 , $O_2S_2 \dots$ equal to the respective values of CD , and OS , O_1S_1 , $O_2S_2 \dots$ represent the pressures $\frac{R}{\epsilon}$ to the variable base

$$\frac{1}{2}A'M', \quad \frac{1}{2}A'M', \quad \frac{1}{2}A'M' \dots$$

(only the first of these bases is shown). The problem to which we address ourselves is to reduce these values of OS to one base, e , where e is constant.

Through A' draw $A'T$ parallel to AS then, OT is the

$$\frac{\text{pressure}}{\text{€}} \quad \text{or} \quad \frac{R}{\text{€}}$$

to the double base AM (not drawn). For subsidiary figure

$$\triangle OAT = \triangle OA'S \text{ for } \triangle AST = \triangle ASA'$$

being upon the same base AS , and between the same parallels $AS, A'T$.

From A let fall a perpendicular on $A'M'$, from the foot of which, along $A'M'$, measure off $2e$. From the free extremity of $2e$ erect a perpendicular, cutting the wall line in the point O_e , draw O_eT_e parallel to OST . To fix our ideas take any one pressure OT , through T draw a line parallel to OA intersecting O_eT_e in a certain point. Through that point and A draw a right line cutting OT in R . OR is the total earth pressure : ϵ on the wall, or $\frac{R}{\epsilon}$ from A to O . We can thus find as many points R as we please. For looking at OT projected on the line O_eT_e we have

$$OT : OR :: AO_e : AO$$

$$:: 2e : AM,$$

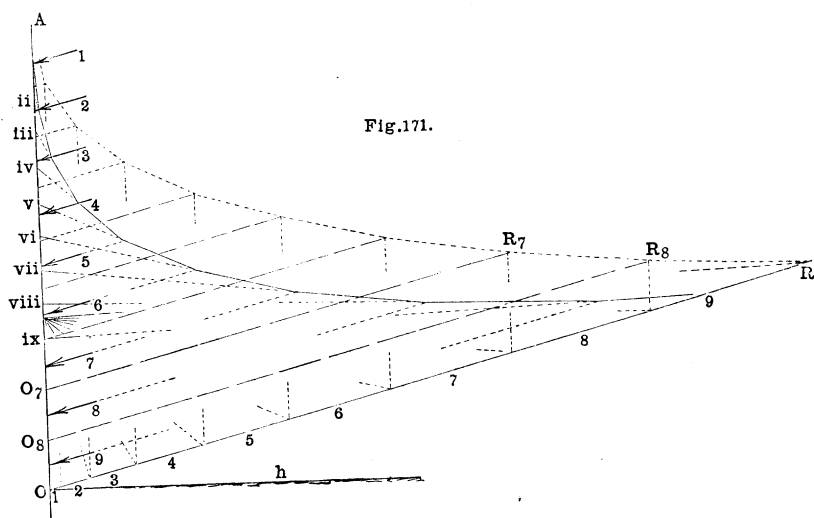
or remembering that product of extremes = product of means

$$OT \cdot \frac{AM}{2} = OR \cdot e.$$

The curve thus formed is evidently a parabola.

257. *Line of Action of Resultant of Pressure upon a Retaining Wall.*—Having thus found a series of pressures : $\epsilon, OR, O_eR_e, O_{e''}R_{e''}, \dots$ (fig. 171), it is evident that the difference between Ok and O_eR_e is the total pressure that acts upon the part OO_e of the wall, whence the following construction in order to find the line of action of the resultant of all these separate forces.

Let AR be the curve of pressures of last article. Divide the area AOR into layers by lines parallel to OR , of such a thickness as that the pressure acting on any individual layer may be conceived as acting at the middle of the layer. Then from R, R_1, R_2, R_3, \dots draw lines parallel to AO , cutting off, on OR , the parts 9, 8, 7, 6 equal to the pressures : ϵ acting at the centre of pressure of their respective layers. Look now upon OR as the line of weight of a force polygon, and placing its pole somewhere in line of the wall, we obtain the usual cord polygon



whose rays extended to the wall line, give the points *ii*, *iii*, *iv* . . . in the lines of the resultants of the pressures

$$\Sigma_1^2, \Sigma_1^3, \Sigma_1^4 \dots$$

The exact position of the centre of pressure of a given layer is afterwards elucidated.

Section II.—Cohesion.

258. *Remarks.*—Cohesion diminishes the pressure exercised by the earth prism, but in most cases it is advisable to neglect it as being altogether too treacherous in its nature to be trusted, yet the atmospheric influences which act upon it behind retaining walls are absent in deep tunnel work, while at the same time it may be lessened, or even destroyed, by the operations there carried on.

259. *Equilibrium of Earth Prism sustained by Cohesion alone.*—Consider a mass of undisturbed earth presenting a face *OA* (fig. 172), steeper than its angle of repose, sustained by no retaining wall, then the force polygon is reduced to three forces, the weight *P* of the prism *AOY*, the oblique reaction *Q*, and the cohesion *K*.

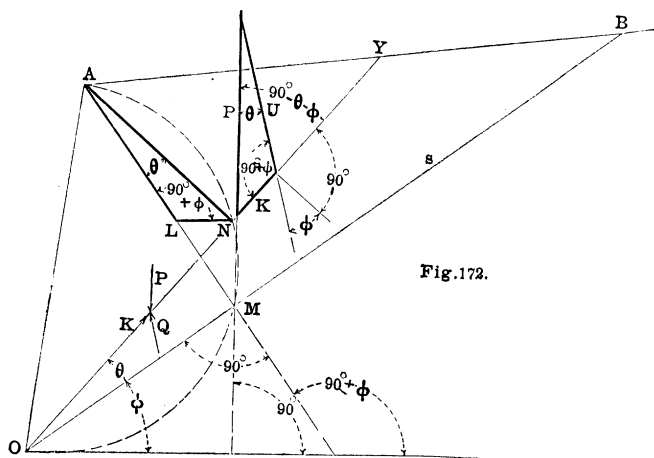


Fig. 172.

260. *Geometrical Construction of the Value of Cohesion for the Prism AOY.*—Take the perpendicular AN upon

$$OY = \frac{P}{\epsilon},$$

then the reducing base evidently is $\frac{1}{2}OY$. Draw the perpendicular AM to the plane of repose, and from N meet it by the horizontal NL . The triangle ANL thus formed is similar to the force polygon PQK .¹

The force polygon PQK has its side K upon the plane of rupture, and the various angles formed by the lines of fig. 172 have been marked so that the student may easily prove for himself the similarity.

We have AN, AL, LN proportional to P, Q, K and

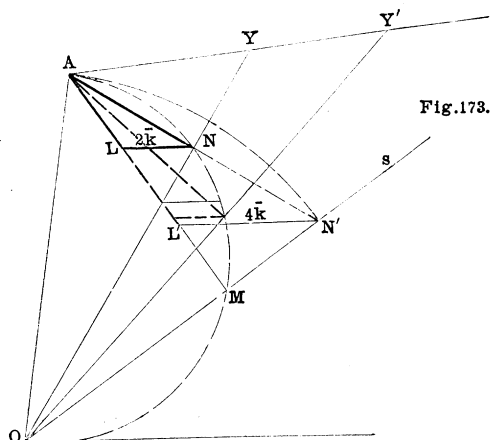
$$K = \frac{OY}{2} \epsilon \cdot NL$$

and it is distributed over OY , whence cohesion per unity of area of

$$\frac{OY \cdot \epsilon \cdot NL}{2OY} = \frac{1}{2} \epsilon \cdot NL = K.$$

If we now give the plane of rupture another direction, we obtain another value of K , and for a certain plane it must reach its maximum value.

¹ U is wrongly inserted for Q in figure.



261. *Geometrical Construction of the Maximum Value of Cohesion.*—Consider now the plane of rupture OY (fig. 173) to pivot round O , then for every direction a length LN is determined, ANO being a right angle; on OA describe a semicircle, necessarily going through the points M and N ; further, AM is a fixed straight line, so that LN will attain its greatest value when N lies in the centre of the arc AM . In this case from elementary considerations

$$\angle AON = NOM \quad \text{or} \quad \angle AOY = \angle YOB.$$

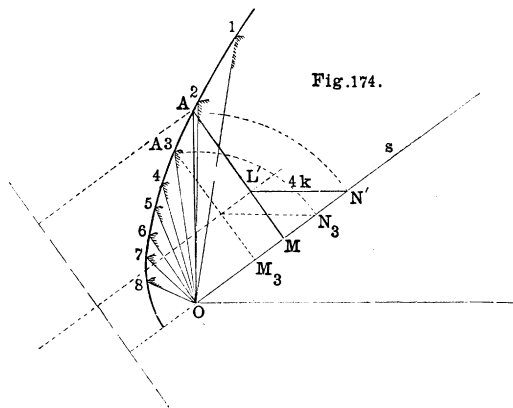
produce AN till it meets AM in N' , then

$$ON' = OA \text{ and } L'N' = 2LN = 4k, \text{ or } k = \frac{1}{4}L'N'.$$

It will be observed that taking O as a centre we can transfer OA to ON at once.

It is worthy of remark that the plane of rupture OY for the maximum value of cohesion coincides with the plane of rupture, OX , in that case (art. 215) where the terrain line is horizontal and $\phi' = 0$.

262. *Parabola of Cohesion (fig. 174).*—It is evident that the unsupported face of earth will approach the plane of repose, the greater the vertical height of OA till, giving it an unlimited height it coincides in that plane, whence the following problem. Knowing the extreme height at which a face OA of given earth, with given inclination, will sustain itself in equilibrium by its



cohesive force, to find the corresponding heights for all inclinations or inversely.

Let it be known, for instance, that a face of given earth will sustain itself vertically for a height $OA = 10$ feet. Then find 4. K for that height by last article, move $4k$, and the perpendicular AM , parallel to themselves along the plane of repose into new positions, as A_3N_3 . From O as centre describe the arc N_3A_3 cutting the perpendicular in A_3 . Join OA_3 , it is one of the faces required.

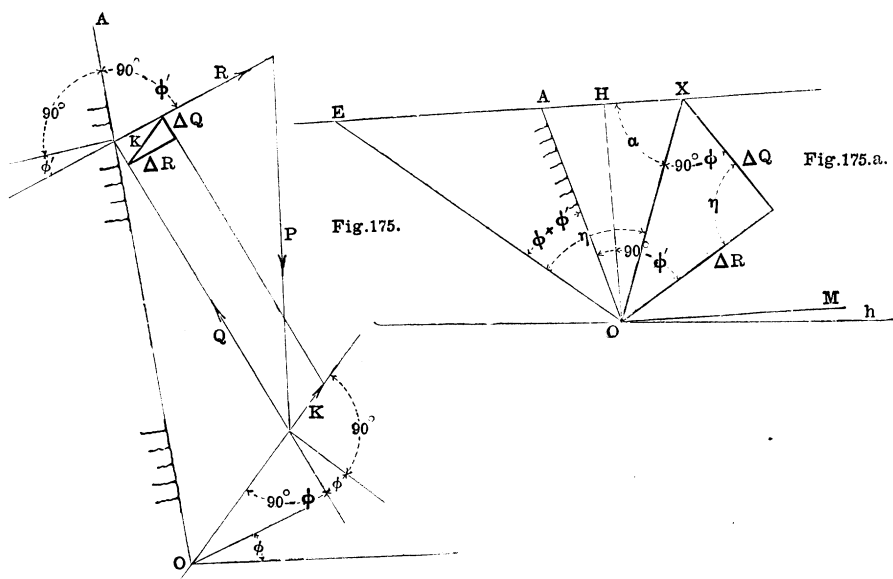
The locus of A is a parabola.

263. *Position of the Plane of Rupture for which the Influence of Cohesion in Diminishing the Earth Pressure is the Least.*—Consider the force polygon PQR , fig. 175, now introduce K into it, and R and Q are diminished by the quantities $\Delta R \Delta Q$.

Notice that the angle $A'R = 90^\circ - \phi'$ and $Q'K = 90^\circ - \phi$.

Consider now the earth prism OAX , fig. 175a, OX being a plane of rupture. As the force of cohesion K is proportional to the length of the plane of rupture (prism unity in length understood), let OX represent the face of cohesion, and draw ON parallel to R , i.e. making an angle $90^\circ - \phi'$ with OA , XN parallel to Q , i.e. making an angle $90^\circ - \phi$ with OX . Draw OE making, as formerly, an angle $EOA = \phi + \phi'$. Mark the angle EOX with η , then $ON = \Delta R$, $XN = \Delta Q$, and the angle $ONX = \eta$, for it is equal

$$\begin{aligned} 180^\circ - XON - OXN &= 180^\circ - [(\phi + \phi' + 90^\circ - \phi') - \eta - (90^\circ - \phi)] \\ &= 180^\circ - \phi - \phi' - 90^\circ + \phi' + \eta - 90^\circ + \phi = \eta. \end{aligned}$$



Let fall a perpendicular OH from O upon the terrain line AX . Mark the angle EXO with α , then we have

$$K = \frac{OH}{\sin \alpha},$$

and

$$\Delta R = \frac{K \cdot \sin(90^\circ - \phi)}{\sin \eta} = \frac{OH \cdot \cos \phi}{\sin \eta \cdot \sin \alpha}.$$

In this expression OH and $\cos \phi$ remain constant, and ΔR is therefore indirectly proportional to $\sin \eta \cdot \sin \alpha$.

Let now the plane of rupture turn round O , so the more we increase η the more we decrease α , whence as $\sin \eta \cdot \sin \alpha$ is greatest when $\eta = \alpha$, ΔR is then the least.

Cohesion exercises the least influence on the earth pressure R when $EO = EX$. Draw OM parallel to EX so in this case the plane of rupture bisects the angle EOM .

264. *General Divergence of Planes of Rupture and Cohesion, and Case of Coincidence.*—In general the plane of rupture OX for the greatest earth pressure is then not the same as the plane

of rupture OY giving the minimum influence of cohesion in diminishing this earth pressure, but the two planes evidently coincide in the case of (215), where $\phi' = 0$ and the terrain line horizontal.

265. *Introducing the Influence of Cohesion into the Construction of the Earth Pressure.*—In the case where the two planes coincide.

This is evidently attained by moving the point O upwards along the line OA for a length equal to the length at which at the inclination of OA the earth would maintain itself in equilibrium by cohesion alone.

In other cases its solution leads to a problem of the third order, and its practical value is by no means commensurate with its difficulty.

In all cases an approximation to the value, if any, which should be given to it, is obtained by moving O upwards along OA , one half of the height found at which it would maintain itself by cohesion alone.

Section III.—Designing Retaining Walls.

266. *Application of the Involution Projected upon EO to one or more Series of Retaining Walls.*—Up till now, in representing an area by a line, we chose some base a , but in combining the forces arising from the weights of a prism of earth and masonry (one unit long) the line representing the thrust of the earth derived from the area of the prism must be multiplied by its weight per cubic unit ϵ , the line representing the vertical downward weight of the masonry derived from its area must be multiplied by its weight per cubic unit μ , but in order that we be able to combine these two lines in the parallelogram of forces so as to obtain their resultant, we have to bear in mind that a representative line is inversely as the length of its base, and that the representative lines of masonry and earth must be directly as their specific gravities, we must therefore have, calling e the earth base, and m the masonry base

$$\frac{\mu}{\epsilon} = \frac{e}{m} \quad \text{or} \quad m = \frac{\epsilon}{\mu} e \quad \text{or} \quad e = \frac{\mu}{\epsilon} m.$$

Return now to equation (4) (art. 213)

$$\frac{R}{\epsilon} = \frac{1}{2} \sin \beta \cdot l^2,$$

add a coefficient of security σ

$$\sigma \frac{R}{\epsilon} = \sigma \frac{1}{2} \sin \beta l^2,$$

and it must be represented by a line, by means of a base

$$e = \frac{\mu}{\epsilon} \cdot m.$$

Let h be the height of the masonry, and $m = \gamma h$, then

$$e = \frac{\mu}{\epsilon} \gamma h, \quad \text{or}$$

Let v be the representative line of the force σR

$$\sigma R = \epsilon e v = \epsilon \frac{\mu}{\epsilon} \gamma h v = \epsilon \sigma \frac{1}{2} \sin \beta \cdot l^2,$$

or

$$\mu \gamma h v = \epsilon \sigma \frac{1}{2} \sin \beta l^2,$$

or

$$v = \frac{\epsilon \cdot \sigma l^2 \sin \beta}{2 \mu \gamma h}.$$

Consider the following proportion

$$\frac{2 \mu \gamma h}{\sigma \epsilon \sin \beta} : l :: l : \frac{\sigma \epsilon l^2 \sin \beta}{2 \mu \gamma h}$$

$$:: l : v$$

or

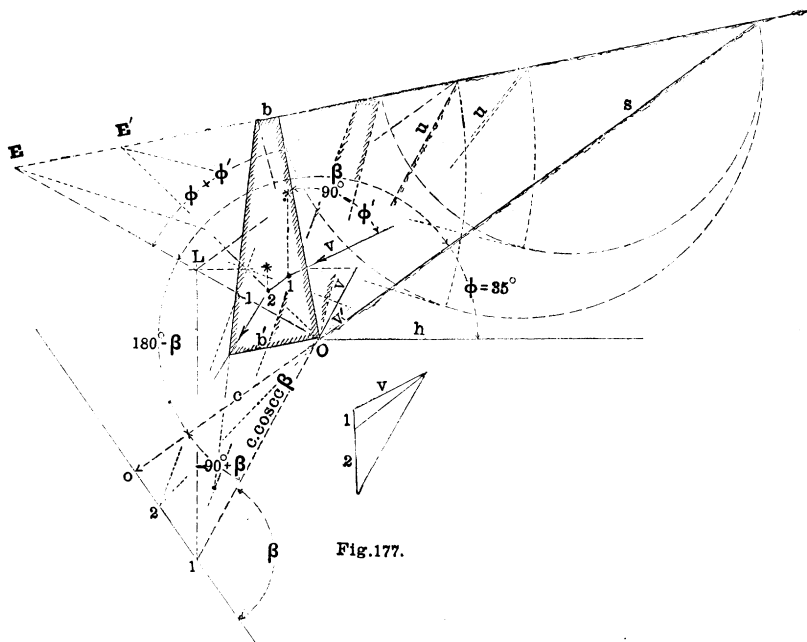
$$\frac{2 \mu}{\sigma \epsilon} \gamma \frac{h}{\sin \beta} : l :: l : v,$$

or

$$f \frac{h}{\sin \beta} : l :: l : v$$

where

$$f = \frac{2 \mu}{\sigma \epsilon} \cdot \gamma.$$



This proportion gives the following construction.

Continue (fig. 177), the line of repose s through O until it is $= fh$ ($fh = c$ on drawing), and through its extremity draw a line $O1$ at right angles. Again through O draw a line at right angles to EO , meeting $O1$ in a point 1, the angle this line makes with fh or c is readily seen to be $90^\circ - (180^\circ - \beta) = -90^\circ + \beta$, and consequently the angle which it makes with the line at right angles to fh is β , whence the length of the line

$$O1 = fh \cdot \frac{1}{\sin \beta} = c \operatorname{cosec} \beta.$$

Join $1L$ and through L draw to $1L$ a line at right angles, this line will cut off v from the continuation of $O1$ backwards.

For example, let the earth to be supported be damp sand weighing 120 lbs. per cubic foot. Let the masonry be slate stone weighing 180 lbs. per cubic foot. Let

$$m = \gamma h = \frac{2}{3} h$$

and coefficient of security $\sigma = 2$, then

$$f = \frac{2}{\sigma} \cdot \frac{\mu}{\epsilon} \cdot \gamma = \frac{2}{2} \cdot \frac{18}{12} \cdot \frac{2}{3} = 1.$$

In this case $fh = h$, and this is the case in fig. 177.

Choosing a breadth b for the top of the wall, we have then a triangular prism of masonry whose height is h , whose vertex is O , whose base is b , this reduced to a base

$$m = \gamma h = \frac{2}{3} h$$

gives the side 1 of the force polygon, and by means of several trials we find an inferior base b' , giving a triangle of masonry which reduced to the base γh gives the side 2 of the force polygon.

But here a small arithmetical operation is preferable to a formal graphical reduction. In this instance

$$\gamma h = \frac{2}{3} h,$$

and the reduced areas of the triangles

$$\frac{h}{2} \cdot b \text{ and } \frac{h}{2} \cdot b'$$

is therefore

$$\frac{h}{2} \cdot \frac{b}{1} \div \frac{2}{3} h = \frac{h}{2} \cdot \frac{b}{1} \cdot \frac{3}{2} \cdot \frac{1}{h} = \frac{3}{4} b.$$

From this force polygon we can trace the line of pressures, and when the line of pressures goes through the toe of the wall we have obtained the correct section of the wall required.

The value of this method lies mostly in the ease with which series of walls, as *I, II, . . .* fig. 177, each wall having a different inclination on the inner side, and each series having different values of

$$h, \frac{\mu}{\epsilon}, b, \text{ \&c.}$$

may be prepared for future consultation.

For a given series within a certain range of interior slope, generally including the whole practical range, it will usually be found that the extensions of the exterior sides of the walls converge closely around a mean point, which thus becomes for

267. Fig. 177 shows how much the quantity of masonry required in a retaining wall depends upon the slope given to its inner surface, affording an elegant explanation of the great efficacy of pitching a steep slope of earth.

268. *Drawing in the Line of Resistance to a Retaining Wall, combined with Prof. HAESELER'S Method of Representing the Earth Pressure.*—We have already found the earth pressure per unit of area at any point O of the wall to be equal to a given line HK .

It is evident that if we lay off the values HK for all points O of the wall, horizontally as ordinates, we obtain an area representing the total earth pressure upon the wall. If the terrain line is in one plane, and the inner line of the wall in one plane AO' (fig. 179) the earth pressure area $AH'K'$ is necessarily a triangle. But if the terrain line is in more than one plane as $AO'O$, the earth pressure area is necessarily an irregular figure. If the inner line of the wall is curvilinear, it must be treated as approximately polygonal, and a value of HK found for each angular point.

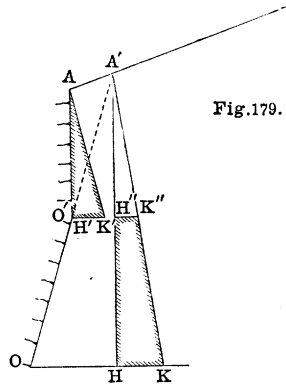


Fig. 179 represents the inner line of a retaining wall, having an angular point at O' , the terrain line AB being in one plane. OO' has been extended to A' , the small triangular prism of earth $AO'A'$ neglected. Then the triangle of earth pressure has been first formed for the part AO' , then for the second part $A'O$. That part of this second triangle may be displaced for convenience so as to be under the first triangle, as in fig. 180.

In order to combination with the weight of the wall, HK itself is not laid down but

$$\frac{\epsilon}{\mu} HK,$$

and then the process of obtaining the line of resistance is as shown in figure.

The retaining wall and earth pressure area are divided into several laminæ 1, 2, 3, 4, 5 of equal depth. The weights of the former 1, 2, 3, 4, 5 act vertically through their centres of gravity G_1, G_2, G_3, G_4, G_5 . The pressures of the latter act at an angle ϕ from the normal upon the back of the wall in a point where the horizontal through the centre of gravity $G'_1, G'_2 \dots G'_5$ of the earth pressure areas cuts the back of the wall.

The common depth of the laminæ has been taken as the base upon which to reduce the areas, *i.e.* shortly, that with the first earth pressure area, being a triangle, half the base is taken as measuring it, and with all the masonry laminæ except the last and with all the remaining earth pressure areas being trapeziums the mean breadth has been taken. The last masonry area of course had to be reduced formally by being transformed into a triangle of the same height and half the base taken. From these lines the force polygon was constructed, and from it the cord polygon.

There is nothing in the construction of the force and cord polygons differing from those of Chap. II. The forces distributed in the plane are $1', 1, 2', 2, 3', 3 \dots$ corresponding to the 1, 2, 3, 4, 5, 6, of fig. 29.

The force polygon forces $1'$ and 1, give the resultant marked $1'1$, cord polygon gives the link $1'1$. $1'1$ and force 2 in the cord polygon combine and give the resultant link $1', 1, 2'$, &c. Where the link $1'1$, combining the pressure upon the first lamina with its weight, crosses the joint a , it gives a point A in the line of resistance. Where the link $1' . 1 . 2' . 2$, combining the pressures upon the first two laminæ with their weights crosses the joint b , it gives a point B in the line of resistance Joining the points $ABCDE \dots$ we have the line of resistance.

In this instance it may be necessary to remark the angle ϕ' has been taken as large as 35° , a value which can only be justified when the earth behind the wall, technically termed backing, has been formed of suitable materials, and well beaten down.

269. *Fluid Pressure behind a Retaining Wall.*—The value of fluid pressure per unit of area and the action it exerts upon a retaining wall, are deduced from the well known properties to be found demonstrated in works on hydraulics, that it is

directly as the depth and weight, and presses equally in all directions. For our purpose the last of these two propositions may be more conveniently, though less generally, enunciated in saying that it acts normally to the opposing surface.

The value of the resultant R of the pressure, and its point of action upon a surface, are usually deduced in works on hydraulics in such a manner as to be an analytical solution of the problems of arts. 151 and 152, where the surface of the water takes the place of the neutral axis.

The student will perceive the analogy without trouble, for in both cases what is wanted is $\Sigma mz^2 = K$ and the point of action of the force Y of art. 122 on the lamina, or surface.

270. *Determination in a Wall Sustaining Fluid. Of Pressure at the Point in the Line of Action of the Resultant Pressure in the Base.*—Fig. 181 is a retaining wall which was intended to sustain an earth pressure only, but which, owing to circumstances, was called upon to sustain the pressure of a fluid mass to the height $h = 13$ ft. 8 in. Equalising the small steps of the back of the wall by the line $OA'A$, from O draw OH' perpendicular to OA' and make $OH' = h$ and join $H'A'$. This is the triangle of fluid pressure upon the wall, *i.e.*

$$\frac{R}{\epsilon} = \text{triangle } OH'A'$$

where ϵ is the weight per unit of the fluid pressure.

This fluid, a mass of fluidised clay, weighed about 100 lbs. per cubic foot, and, taking, as we have all along done, the length of our prism as unity, cubic contents = area \times 1. The wall was built of rubble basalt, with a facing of 1.5 feet of solid stone weighing about 180 lbs. per cubic foot. The rubble consequently weighing $180 \times 0.4 = 72$ lbs. per cubic foot. We will reduce our fluid pressure triangle, having a density of 100 lbs. per unit, to a triangle having a density of 72 lbs. per unit, whence the base of the new triangle must be

$$\frac{\epsilon}{\mu} h = \frac{\epsilon}{\mu} OH' = \frac{100}{72} OH' = OK.$$

This has been obtained by a small graphical multiplication, shown in fine dotted lines, with which the reader is now perfectly

familiar. The measure of the area of this triangle is half its base OK to the height h taken as the reducing base, which we will reduce to a base the height of the masonry OA (very nearly), whence the base shown by a thick line in the measure of the triangle. The central breadth of the trapezium, 1, of masonry is the measure of that trapezium. The measure of the cut stone, 2, in front is 2.5 times its breadth ($2.5 \times 72 = 180$). The rubble foundation, 3, has to be reduced to the base OA (shown in fine dotted lines).

The fluid pressure acts normally to the back of the wall at a height from $O = \frac{1}{3}h$. We can now form our force and cord polygons in the ordinary way and obtain the point X of the action of the resultant pressure upon the base.

271. Distribution of Pressure upon the Base of a Retaining Wall.—We have already (122) brought prominently before the student the unequal distribution of pressure upon any joint of masonry arising from the deviation of the point of action of the resultant from the centre, which inequality of distribution must evidently hold good at the base. Hence the necessity of having a foundation so hard that this inequality in the pressure is a matter of indifference, or such an enlargement of the last courses of the base, technically called the footings, as to bring the point of action of the resultant within a safe distance of the centre.

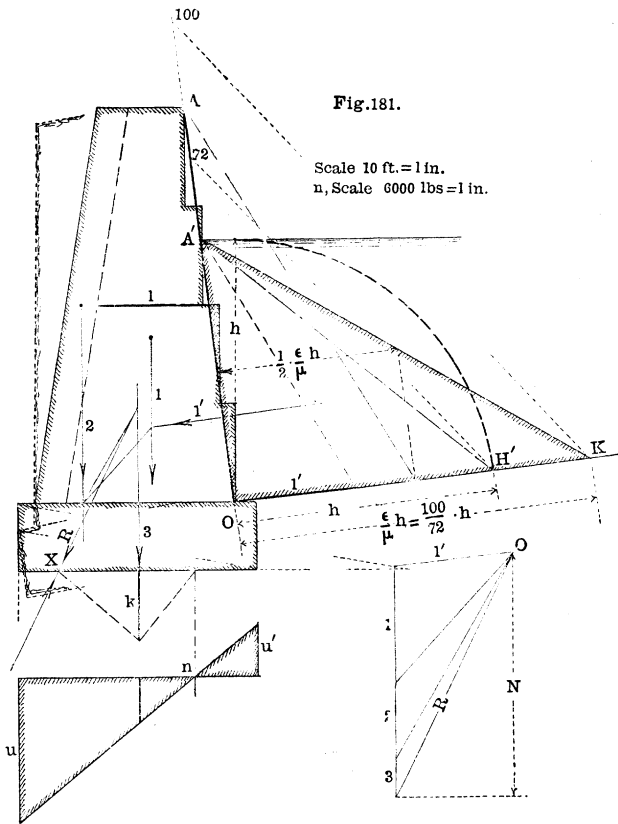
Let us now investigate the distribution of pressure upon the foundation of this wall arising from the foregoing fluid mass, uniting with our graphical solution our arithmetical calculations.

The weight of the fluid pressure triangle

$$\begin{aligned} OA'H' &= \text{area} \times 1 \times 100 \text{ lbs.} \\ &= \frac{1}{2} \times 13'.66 \times 13.66 \times 100 = 9330 \text{ lbs.} \end{aligned}$$

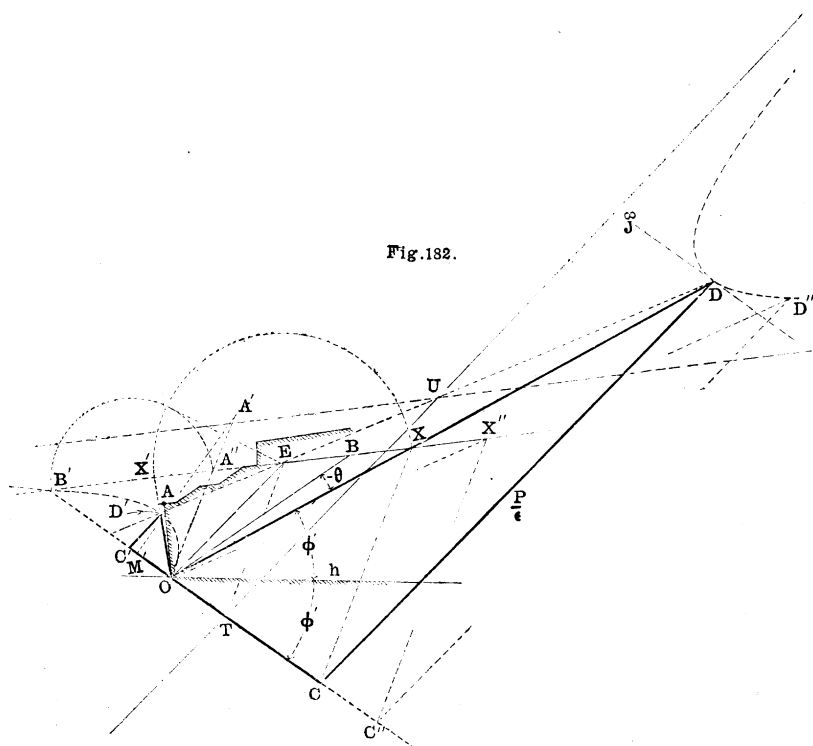
Lay this off from O on the line $1'$ of the force polygon to any convenient scale, and through it draw a vertical line. Resolving R into horizontal and vertical we find the vertical or normal component $N = 18,650$. Let now, as in art. 151, the central breadth of the trapezium measure the area. It will do so necessarily to the breadth of the base of the wall, or 12.5 feet.

$$\frac{18,650}{12.5} = 1,555 \text{ lbs.} = \text{pressure per square foot at the centre of}$$



the wall. Lay this central breadth off in its appropriate place (fig. *c*), and through its free extremity, and through the antipole of point action of *R* draw a line. This line will cut off *u* and *u'* respectively upon the toe and heel of the base, as they are technically termed, *u* measuring 4,600 lbs. per square foot of compression upon the base of the wall and upon its foundation, or 2.05 tons, and *u'* 1,600 lbs. of tension upon the lower part of the back of the wall, or $\frac{2}{3}$ ths its capacity to resist tearing. For simplicity of diagram, we have neglected the earth pressure acting upon 3, and which is sufficiently large to remove the line of action of *R* so far outwards as to cause it to pass through the toe of the wall.

272. *Example of a Bad Foundation.*—As a good foundation, unlike a well-designed structure, teaches us nothing, whereas a bad foundation is full of instruction, we have, in the above, chosen the case of a wall which at one time fell within our observation, as affording an apt illustration of the fate which befalls a structure reared on a treacherous foundation. This massive quay wall was founded upon a base of clay, and in excavating for a foundation the rule adopted was, that when the underlying clay bore the weight of a man so that the impact of his feet upon it left but the impress of his hobnailed boots, it was fit to build upon. Among the consequences of the observance of this eccentric rule, and of the fluid pressure bearing directly upon the wall, were the sinking of the toe, as shown in figure, and the gradual overturning of a costly structure.



273. *Construction of the Minimum Value of the Passive Resistance of Earth Pressure.*—The method of determining the minimum value of the passive resistance of earth pressure is analogous to the method of determining the maximum value of active pressure. In that of active pressure we use the plane of repose, in that of passive resistance we use the reflection of the plane of repose. In active pressure, the line OE makes an angle $\pm (\phi + \phi')$ with that of the wall. In passive pressure the line OE makes an angle $\pm (\phi + \phi')$. In the same manner the point X is found, but is also located on the other side of E . Fig. 182 shows the construction, while we bear in mind arts. 198, 199, 200. In this figure the construction of the minimum passive resistance force polygon is shown with the same letters as formerly with that of active pressure, and conceiving for a moment the possibility of an overhanging earth prism $B'OA''$ the real earth prism $OA''B$ removed, which let us call the complementary earth prism, the accented letters belong to the construction of the active pressure of this prism.

The student will grasp the points of analogy from the figure more clearly than by a detailed description— CD is the passive resistance. We observe at once how much greater the passive resistance can be than the active pressure.

We shall immediately see that the points D, D' are on opposite branches of the hyperbola, but on the same diameter, so that, from this circumstance, we can condense the construction one half, for DD' being bisected by the centre U of the hyperbola $D'U = DU$, whence $CD = UT + (UT - C'D')$ or $2UT - C'D' = CD$, so that by finding, as formerly, the active pressure $C'D'$ of the complementary earth prism and the centre of the hyperbola, we have given to us the passive resistance.

274. *The Point D of the Ordinate of Passive Resistance lies on the Opposite Hyperbola to the Point D' of the Ordinate of Active Pressure.*—The demonstration is the same as formerly, except that in fig. 183 the double point D of the homologous circle in the involution with centre J is taken on the opposite side at D_1 . With this substitution, the demonstration is the same as in arts. 199, 200, we will therefore not formally recapitulate it, but to assist the conception of the student, we have in fig. 183 carried



276. *Fluid Pressure upon Overfall Weir.*—In this case, fig. 188, the pressure triangle has its vertex on the surface of the

A diagram showing a stepped profile. The profile consists of a flat base, followed by a vertical rise to a platform, then a horizontal gap, then another vertical rise to a higher platform, and finally a horizontal top surface. Two vertical dashed lines are drawn: one through the first vertical rise and one through the second. Point A is on the first vertical rise, point B is on the second vertical rise, point C is on the horizontal top surface, and point D is on the horizontal top surface. The profile is shaded with diagonal lines.

water above the wall. The method of finding points in the line of resistance depends on art. 217. The point in that line for the joint 4, 5 is alone drawn in. The point A is the pole of the masonry force polygon with which to form the cord polygon. In the same manner B has been chosen as the pole of the water force polygon.

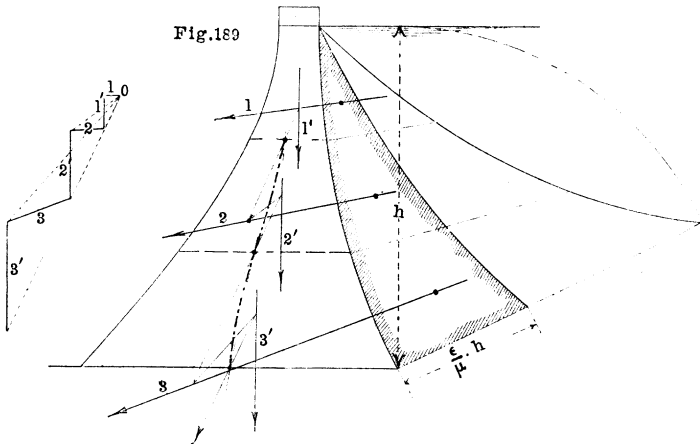
277. *Relation between the Centre of Gravity of a Trapezium, and a Certain Pole and Polar.*—From a consideration of preceding articles, we perceive that if we take the point of intersection of the non-parallel sides of a trapezium as pole, to the height of the trapezium, considered as the side of a rectangle, the pole is a line parallel to the parallel sides and passing through the centre of gravity of the trapezium. For instance, referring to fig. 90, take one of the trapeziums as that which has the parallel sides u and u_1 , the point $n, 1$ is pole, the polar line $\bar{1} 1$, parallel to u and u_1 , passes through the centre of gravity of that trapezium, and it is the same thing for instance to find ${}^w R_1^t$ (fig. 188) by means of the antipolar of the surface n of the water or by means of the centre of gravity of the trapezium $ABCD$.

278. *Profile of a Reservoir Wall.*—A modern reservoir wall is built to fulfil certain conditions.¹ When the reservoir is full empty that the pressure on the exterior interior side do not exceed a certain amount, that the interior exterior side be not put in tension, although greatly relieved of pressure. These conditions require the lines of resistance to be within the central third of the wall and near its exterior interior limit.

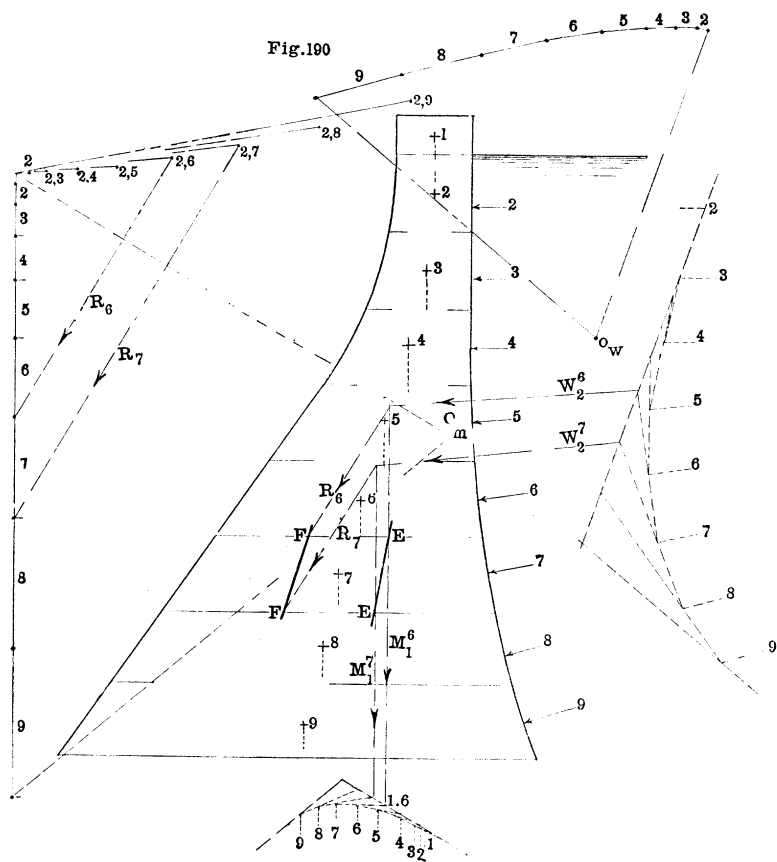
Such a wall can be designed *tentatively* by means of the preceding methods, examining as we proceed from layer to layer downwards the states of exterior pressure u by means of the method employed in art. 271, fig. 181, and rectifying.

¹ First indicated by M. DE SAGILLY, *Annales des Ponts et Chaussées*, 1853, and perfected by M. DELOCRE, *Annales des Ponts et Chaussées*, 1866.

Fig. 190 is such a wall, and the complete construction of two points $F' E'$ in the line of resistance for the reservoir full is given, and requires no explanation. Or Häsele's force polygon may be employed, figs. 180 and 189.



In practice it will be well to consult while designing, the types of reservoir walls by Krantz in "*Étude sur les Murs de Reservoirs, par J. B. Krantz.*" Paris. 1870.



CHAPTER VIII.

THE TUNNEL.

Section I.—The Symmetrical Tunnel.

279. *Pressures acting upon a Tunnel Arch in Earth having a Horizontal Terrain Line.*—We consider that the earth pressure upon the arch is normal, that is, $\phi' = 0$, and the terrain line being horizontal, the case assimilates to that of (255) fig. 169.

Dividing then the arch into convenient portions 1, 2, 3, 4, we consider each portion as a retaining wall, *e.g.* portion 1, and the pressure per unity H, K , found by method of (214) fig. 168, the central point 1 of the extrados of the portion 1 being the point for which H, K , is found and a close approximation to the average pressure per unit of area exposed. The point 1 has been moved upwards along the line produced $\overline{1A}$, of the extrados, to the point O_1 , half the extent of the corresponding ray $S1$ of the parabola of cohesion, cohesion being taken into account to half its value. The process of (215) is then repeated for the other parts 2, 3, 4, and as shown in plate VI. HK is multiplied by the area of the part for which it has been found reduced to a convenient base, and united with the downward pressure of the weight of the masonry of that part properly reduced, shown by ,, on the figure.

280. *Line of Pressures in a Tunnel Arch.*—We now call in the aid of Heuser's problem, remembering that the arch is symmetrical and symmetrically loaded, so that the pressure generated on the crown must be horizontal. A closely approximative line of pressures has been found and drawn in, in our example.

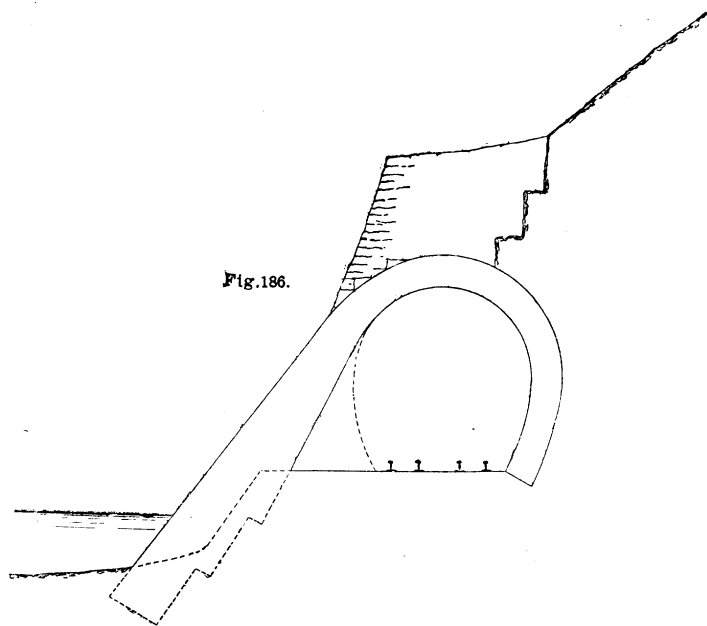
Having then a line of pressures, we now investigate a number of its cross sections, after the manner of an arch, whether it be of iron or stone.

Section II.—Unsymmetrical Tunnel.

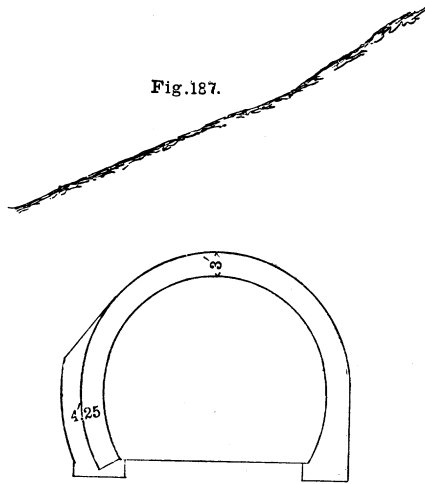
281. Unsymmetrical tunnels may be necessary in sidelong ground, and for the resolution of the pressures acting upon them graphic and statical methods are absolutely essential, and the method of arriving at the form to be given is necessarily tentative, and would be arrived at after various trials. Designing first a symmetrical tunnel, finding the forces acting thereon, and endeavouring to pass a line of pressures through it, we are led to design a more appropriate form, for which we must again find the forces and again endeavour to pass a line of pressures through it.

In finding the forces acting upon the tunnel arching in sidelong ground, the involution in EO , fig. 166, is the most useful, and on OB for the part near the summit.

282. The necessity of an unsymmetrical form in some situations is shown in fig. 186, the original design for the tunnel of



the Call-Trier line of the Rheinische Eisenbahn, only departed from on finding that the nature of the river bed required so deep foundations that it was more economical to remove the site of the tunnel farther under the hill, when, although a symmetrical form was adopted, the extrados required to be adapted to the pressures as shown for one place in fig. 187.¹



We can only obtain an approximate section for an unsymmetrical tunnel by a number of trials, beginning by drawing in the most axial pressure line to a symmetrical form, then modifying the section to suit the pressure line and beginning again anew, and repeating till a suitable form has been obtained. A first symmetrical section with its pressure line, and the final form with its pressure line, have been given by Prof. Wilhelm Ritter in his valuable little work, *Die Statik der Tunnel Gewölbe* (Berlin, 1879), from part of which fig. 191 has been reduced.

283. *Tunnel Sole*.—The resultant pressures e' (fig. 185) through the ends of the tunnel ring, whose values are found from the force polygon, may exercise a pressure upon the

¹ We have to thank the Direction of the Rhenish Railway Company for the spontaneous courtesy with which they supplied the information and drawings from which the two here given have been selected.

To face page 338.

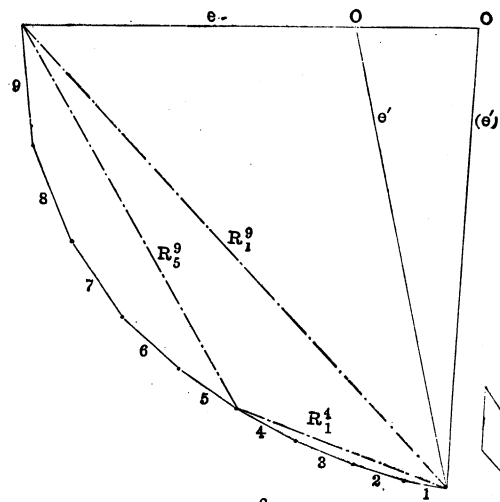
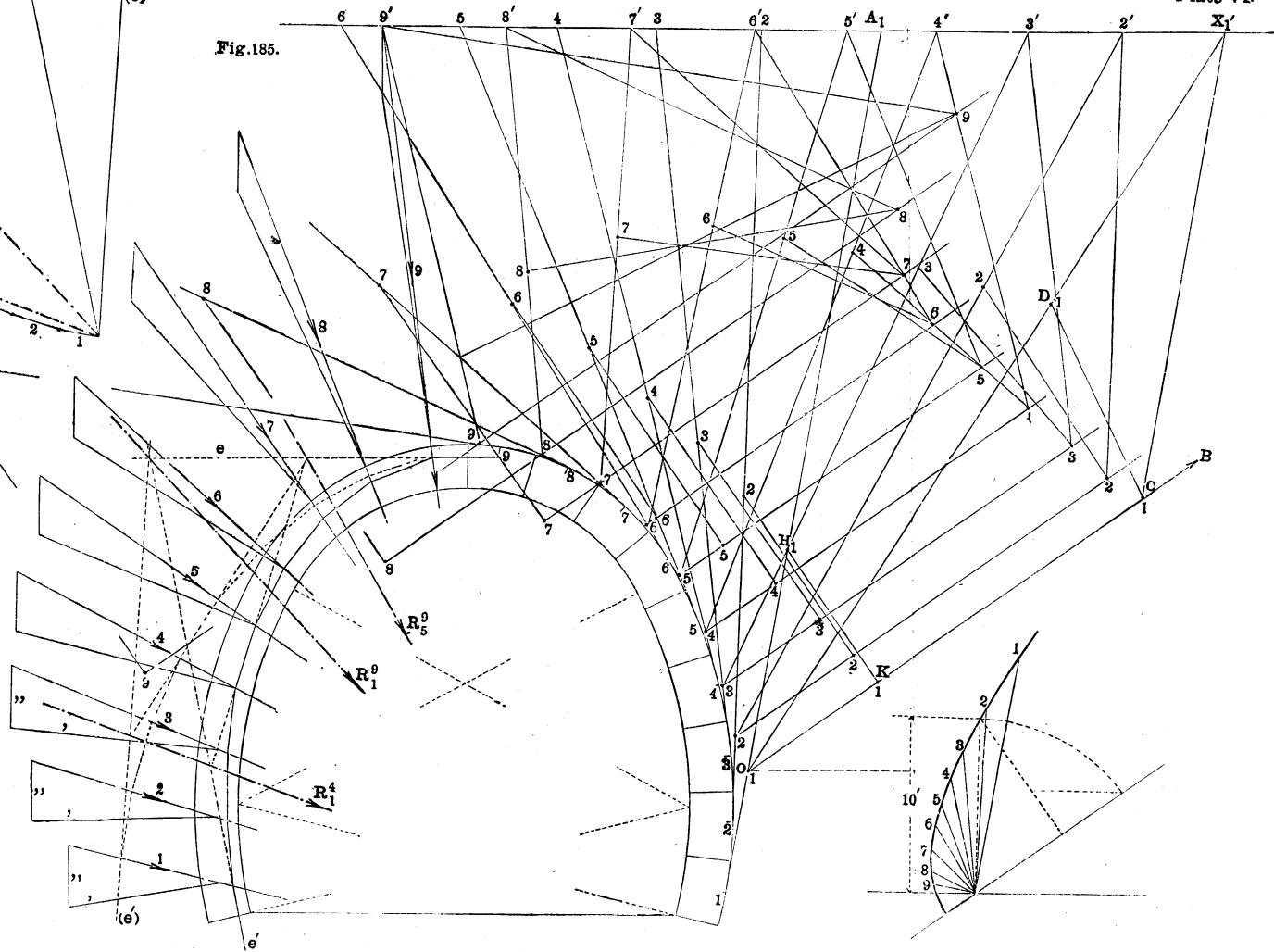


Fig. 185.

Plate VI.



CHAPTER IX.

PROJECTIVE GEOMETRY.

Section I.—Anharmonic Ratio.

I. *Principle of Signs in Geometry.*—If a straight whose extreme points are A and B be indicated by AB when measured in the direction from A to B , and by BA when measured in the opposite direction, viz. from B to A , then AB is to be regarded as $= -BA$, or in other terms

$$AB + BA = 0.$$

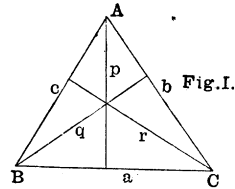
If A and B be any two points on a line, and P any third point on the same line, arbitrarily chosen either between the points A and B or external to them, then having regard to signs

$$AP - BP = AB.$$

The point P , according to its position, divides the line AB internally or externally, and the difference of these two segments is equal to the length of the line.¹

II. *Lemma. Expression for the Area of a Triangle in Terms of Two Sides and their Included Angle.*—

The area of a triangle is equal to the product of two sides multiplied by the sine of the included angle. For fig. i. Let A, B, C , be the three angles of a triangle, a, b, c their opposite sides, p, q, r , the three perpendiculars on the latter from the opposite vertices, then



$$2 \times \text{area} = ap = bq = cr.$$

Now

$$p = b \sin C = c \sin B$$

$$q = c \sin A = a \sin C$$

$$r = a \sin B = b \sin A$$

whence

$$2 \times \text{area} = ab \sin C = bc \sin A = ac \sin B.$$

¹ See Townsend's *Modern Geometry*, vol. i. chap. v.

III. *Corresponding Points in Two Transversals, cutting the same Pencil of Rays.*—Consider any two straight lines u and u' in one plane, fig. ii. Any ray led through a point O in their plane encounters u and u' in two corresponding points A and A' .

Let this ray be mobile, and pivot around O , then as it turns round, the points A and A' move simultaneously along the lines u and u' . Four positions of this ray as it turns mark four points as A, B, C, D on u , and four corresponding points A', B', C', D' on u' , u and u' are called transversals of the pencil of rays.

IV. *The Infinitely Distant Point on One Transversal and its Correspondent Point on Another.*—As the ray turns round, the point A will gradually recede along the line u , until the ray having become parallel to u , the point A becomes situated at an infinite distance on the line u . It is then called the point I at infinity, and its correspondent on another transversal as u' is called I' .

In the same manner, the straight u' has a point J' at infinity whose correspondent on u is called J .

V. *The Constant Rectangle in Two Transversals of the same Pencil of Rays.*—Consider, from a point O , four rays a, b, c, d , meeting the straights u and u' in two corresponding series of points A, B, C, D upon u and A', B', C', D' upon u' , the segments cut off from the two lines are connected by means of the constant rectangle

$$JO \cdot I'O = JA \cdot I'A' = JB \cdot I'B' \dots$$

J, I' and O being as already defined.

For the similar triangles $OAJ, OA'I'$, fig. ii., give

$$JA : JO :: I'O : I'A';$$

likewise the similar triangles $OBJ, OB'I'$, give

$$JB : JO :: I'O : I'B',$$

whence

$$JO \cdot I'O = JA \cdot I'A' = JB \cdot I'B',$$

that is, the rectangle $JA \cdot I'A'$ is constant whatever be the position of the ray.

Va. *Definition*.—The ratio

$$\frac{AC}{BC} : \frac{AD}{BD}$$

of four points $A B C D$ on a line is called the double (Germany) or anharmonic ratio (France, England).

VI. *The Double or Anharmonic Ratio of Four Points in all Transversals of the same Pencil of Rays is Constant*.—Let a, b, c, d signify at the same time the individual rays of a pencil with centre O , and the distances OA, OB, OC, OD from O to any transversal u , then lemma (ii),

$$\begin{aligned} \frac{AC}{BC} : \frac{AD}{BD} &= \frac{a \cdot c \cdot \sin \hat{a}c}{b \cdot c \cdot \sin \hat{b}c} : \frac{a \cdot d \cdot \sin \hat{a}d}{b \cdot d \cdot \sin \hat{b}d} \\ &= \frac{acbd \sin \hat{a}c \cdot \sin \hat{b}d}{adb c \sin \hat{a}d \cdot \sin \hat{b}c} = \frac{\sin \hat{a}c \cdot \sin \hat{b}d}{\sin \hat{a}d \cdot \sin \hat{b}c}. \end{aligned}$$

Here the lengths of the rays a, b, c, d , disappear from the expression for the double ratio, which contains only a relation between the sines of the angles formed by the rays, whence it holds true for all transversals of the same pencil, and the proposition enunciated has been proved.

VII. *The Anharmonic Ratio of All Pencils of Four Rays passing through the same Four Points of a Straight Line is Constant*.—In the expression in VI. for the anharmonic ratio of four points in a line in terms of the sines of the angles formed by a pencil of rays passing through them, the centre O of the pencil is indeterminate, wherefore, All pencils of four rays from perfectly arbitrary centres of space which cut their common transversal in the same points A, B, C, D have the same anharmonic ratio.

Remark.—The equality of anharmonic ratio on different straights, or

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$$

is generally represented thus

$$(A B C D) = (A' B' C' D').$$

VIII. *Given Three Points A, B, C, figs. iii. and iv., on a Straight u to determine with the Help of the Infinitely Distant Point a Fourth Point D in the same Straight so that the Anharmonic Ratio of A, B, C, D, may have any given Value $\pm \gamma$.*—Let a transversal w , fig. ii., be taken parallel to the ray OD , then the ray OD meets the transversal w at infinity, and the equation of equianharmonism becomes

$$\frac{A'C'}{B'C'} : \frac{A'D'}{B'D'} = \frac{A,C'}{B,C'} : \frac{A,D'}{B,D'} = \frac{A,C'}{B,C'} : \infty = \frac{A,C'}{B,C'}$$

D , being at infinity and $\frac{\infty}{\infty}$ being = unity. We have hence the following method. Through C , figs. iii. and iv. draw any transversal w , and upon it from C lay off two segments CA , CB , on the same side of u if γ is positive (fig. iii.) on opposite sides, if γ is negative (fig. iv.) so that

$$\frac{CA}{CB} = \pm \gamma.$$

Join AA , BB , and let O be their point of intersection, then a parallel d to CA , led through O will encounter u in the required point D .

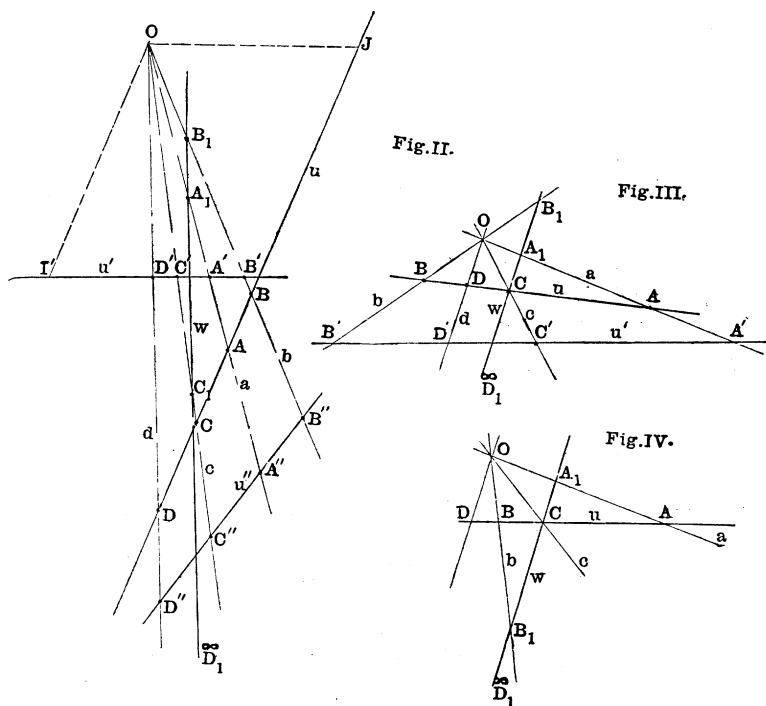
For the line CA , corresponds to the transversal w of fig. ii., and the point D , upon it is at infinity, O is the centre of a pencil of rays a , b , c , d , of which w , u , or any other u' are transversals.

IX. *Definition of Harmonic Ratio, Perspective, and Projective.*

1. *Harmonic Ratio or Section.*—When $\gamma = -1$, giving $CA = -CB$, the double ratio is named harmonic ratio, and is a case of so much importance as to require separate consideration.

2. Two or more transversals cut by the same pencil of rays give two or more series of points *in perspective*, necessarily having the same anharmonic ratio. Two or more pencils of rays passing through the same points in one transversal are *in perspective*, and have necessarily the same anharmonic ratio.

3. Two or more rows of points, having the same anharmonic ratio, but so placed that one pencil of rays cannot pass through both, are *projective*. Two or more pencils of rays having the



same anharmonic ratio, but so placed that they cannot intersect in the same points in some transversal, are *projective*.

X. *Fundamental Anharmonic Ratios*.—The anharmonic ratio of four points or rays is preserved, when in changing the order of two points we change the order of the other two, *i.e.*

$$(ABCD) = (BADC) = (CDAB) = (DCBA)$$

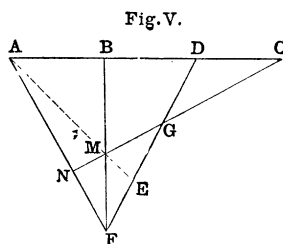
or

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{BD}{AD} : \frac{BC}{AC} = \frac{CA}{DA} : \frac{CB}{DB} = \frac{DB}{CB} : \frac{DA}{CA}.$$

Take for centres of projection, fig. v., *M, A, F*, consecutively; then as anharmonic ratio is preserved unchanged in projection (vi. and vii.)

$$M(ABCD) = M(EFGD) = A(MNGC) = F(BADC).$$

In the same manner we may prove the equianharmonism of the other forms, or we may perceive the equality of the ratios from their statement in the second line.



XI. *Four Points A, B, C, D of a Line determine Twenty-four Anharmonic Ratios, but amongst which are only Six different the One from the Other.*—If we represent for example $(ABCD)$ by a , we have

$$(ABCD) = (BADC) = (CDAB) = (DCBA) = a,$$

$$(ACDB) = (CABD) = (DBAC) = (BDCA) = \frac{1}{1-a},$$

$$(ADBC) = (DACB) = (BCAD) = (CBDA) = \frac{a-1}{a},$$

$$(ABDC) = (BACD) = (DCAB) = (CDBA) = \frac{1}{a},$$

$$(ACBD) = (CADB) = (BDAC) = (DBCA) = 1-a,$$

$$(ADCB) = (DABC) = (CBAD) = (BCDA) = \frac{a}{a-1}.$$

Whence it follows, that if one of the six anharmonic ratios is given, the remaining five are determined.

Section II.—Homology.

XII. *DESARGUE'S Theorem regarding Homologous Triangles.*—Consider a triangle upon a plane α (fig. vi.), whose vertices are A, B, C , and whose sides respectively opposite these vertices are a, b, c . Let rays proceed from a point O to the triangle, so that the triangle may be considered the section of a triangular pyramid whose vertex is O , or, let us consider O a luminous point, casting a shadow of the triangle upon a plane α' , thus forming upon α' a new triangle, whose vertices are A', B', C' ,

Fig. VI.

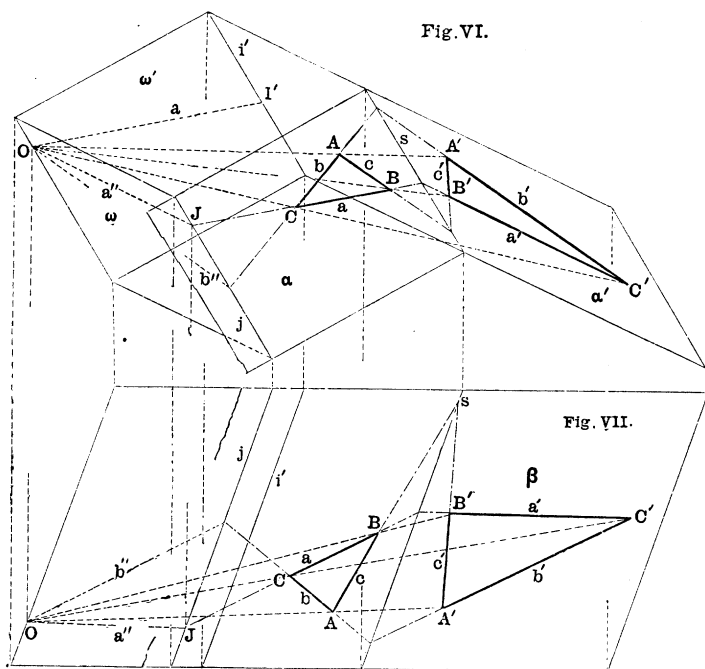


Fig. VII.

and sides a' , b' , c' , then a ray led from O through A passes through A' . In the same manner OB passes through B' , OC passes through C' , then Desargue's theorem is: If lines through each pair of corresponding vertices AA' , BB' , CC' meet in a point O , then corresponding sides of the two triangles aa' , bb' , cc' , meet (two and two) on the line of intersection s of the two planes a and a' and reciprocally; for instance the point O , the lines a and a' lie in the same plane, that plane having a for its trace on the plane a , and a' for its trace on the plane a' , and if we conceive this plane generated by the ray $OC'C'$ moving along a and a' but fixed at O , then when this ray arrives at the line s it will have generated the lines a and a' along the two planes a and a' , and it is evident that a and a' meet in a point on the line s . In the same manner can we perceive that the sides b , b' and c , c' meet in other two points on s .

Let these two triangles be projected orthographically upon a plane β , fig. vii., then, as the projection of a straight line is a straight line, that of a point, a point, we obtain on the plane

β two triangles having the lines joining their corresponding vertices meeting in a point O , and the corresponding sides aa' , bb' , cc' meeting in three points upon a straight s the projection of s in fig. v.

XIII. *Another Enunciation of DESARGUE's Theorem.*—If the three summits A , B , C of a variable triangle, fig. vi., travel along three fixed straight OA , OB , OC which concur in a point O , and if two of its sides as a and b turn around two fixed points (a, s) and (b, s) , then its third side will turn around a fixed point (c, s) in the same straight line with the two points (a, s) and (b, s) . For, fig. vi., $A'B'C'$ is another position of the variable triangle ABC fulfilling these conditions, and if we turn the plane a' round the straight s as around a hinge, we would have the plane a' by its new intersections with the three planes Oa , Ob , Oc forming new triangles fulfilling the conditions.

XIV. *The Infinitely Distant Straight Line the Locus of all Infinitely Distant Points which are situated in the Same Plane.*—

Let a plane ω pass through O , fig. vi., parallel to the plane a' giving the straight j as its trace on the plane a . Let J be the intersection of the line a with the line j , then we have a plane Oa' and two transversals a and a' upon it, and J is the correspondent of the infinitely distant point J' of the line a' and from the parallelism of the planes ω and a , OJ is parallel to a' . In the same manner the point (j, b) on j is the correspondent of the infinitely distant point of the line b' , and the line $O(j, b)$ is parallel to b' , and generally, the straight j is the correspondent of the straight j' at infinity upon which all the points at infinity of the plane a' are situated, whence the conclusion, All points at infinity on a plane are situated ideally on a straight at infinity, this straight being on the same plane.

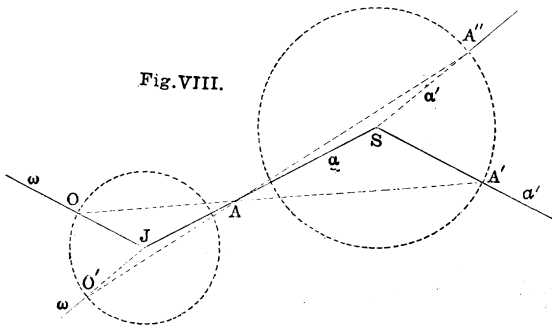
Let again a plane ω' pass through O parallel to a , giving the trace i' on the plane a' , the point (i', a) or I' is the correspondent of the infinitely distant point I of the line a , and OI' is parallel to a . In the same manner the point $(i' b)$ (not drawn in figure) is the corresponding point of the infinitely distant point of b , and $O(i' b)$ is parallel to b , and generally the line i' is the correspondent of the infinitely distant straight i lying on the plane a .

Project these lines j and i' orthographically on the plane β , fig. vii., also OJ , OI' . . . and since the projection of parallel lines are parallel, that

1. The correspondents of all points at infinity of a figure on the plane a' lie on the line j .

2. The correspondent of the point at infinity of a line such as a' is found on the line j and at a point on that line obtained by drawing a ray from O parallel to a' .

XV. *Fundamental Theorem of Homology.*—Instead of conceiving the figures on a and a' as projected orthographically on a plane β , we can conceive the two planes a and ω turning round the lines s and j as on two hinges, but always retaining their parallelism, then the point O would retain its position on the plane ω .

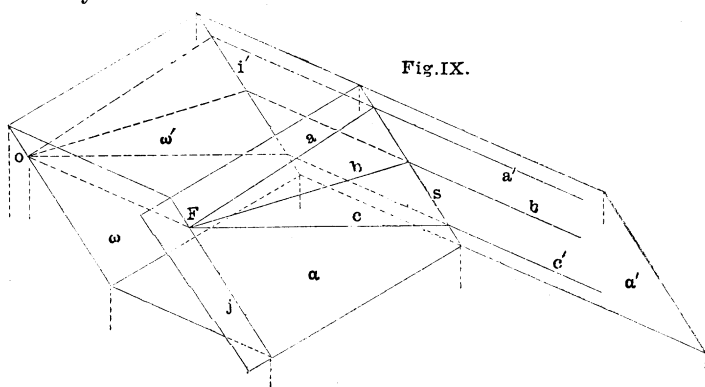


Let fig. viii. be a section of the three planes a , a' ω by a plane perpendicular to them. Let O , A , A' be the orthographic projection of the centre of projection O , fig. vi., and of a pair of corresponding points A and A' , and the points J and S the traces of the lines j and s . Let ω and a' retaining their parallelism turn round the points J and S , then O and A' describe circles. Let O' and A'' be any new position of O and A' , the new ray $O'A''$ passes through the same point A as the former ray OA' , for

$$OJ : JA :: SA' : SA$$

in all positions of the planes. This holds good when ω and a' have been so turned round that the three planes a , a' , and ω are all situated in one plane, whence if the vertices of two figures in one plane taken two and two meet in a point, the corresponding sides meet in pairs on a straight line.

XVa. Remarks.—Fig. vi. being in reality on one plane, that of the paper, gives actually a pair of homologous figures, being already an orthographic projection of a figure in space; but often a clearer view of the derivation of the one homologous figure from the other is gained by retaining in the mind the conception of distinct planes, and we will therefore use it when necessary.



XVI. Projections of Parallel Rays.—Let there be a pencil of rays, a, b, c , fig. ix., on the plane a , whose focus is on the line j , at the point F , then their projections on a' are the traces of the planes, Oa, Ob, Oc, Od , and as these planes pivot round OF as a hinge, and the line OF is parallel to the plane a' (being on ω) the traces of these planes on a' are lines a', b', c' , parallel among themselves and to OF , and F is thus the projection of the focus F' at infinity of the rays a', b', c' .

Inversely we may suppose the pencil of parallel rays given on a plane a or a' , and from them find the pencil of rays on a' or a and the focus on i' or j .

XVII. Parallel Rays meet in a Point at Infinity.—This is a simple corollary from XVI.

XVIII. *Pencils of Parallel Rays on One Plane meet on the Infinitely Distant Straight in as many Points as there are Separate Pencils.*—For two or more pencils of parallel rays on a plane, as a' , may be projected into as many pencils of convergent rays whose foci all lie on the line j ; whence each pencil of parallel rays in one plane, as a' , meets in a point on the straight j' at infinity situated on a' .

XIX. *Plane at Infinity.*—An extension of the foregoing reasoning leads to the ideal plane at infinity, in which lie all the straights at infinity, and consequently all points of space at infinity may be considered as appertaining to the same plane at infinity.

XX. *Definitions of a Complete Quadrilateral and a Complete Quadrangle.*—A complete quadrilateral comprehends the three common quadrilaterals which can be formed by the three possible combinations of the sides and diagonals of a common quadrilateral extended till they meet, and contains therefore six lines. Thus fig. xi., $ABCDEF$ is a complete quadrilateral. AC , BD , EF , are its three diagonals.

A complete quadrangle is the same as a complete quadrilateral, but in considering the dual properties of geometrical figures, we speak of a quadrangle, when we proceed from its given summits to deduce the properties of its lines, and we speak of a quadrilateral when we proceed from its given lines to deduce the properties of its summits.

XXI. *Theorem upon the Projectivity of Complete Quadrilaterals.*—If two complete quadrilaterals, $ABCD$, $A'B'C'D'$, possess the property that the sides AB , $A'B'$; BC , $B'C'$; CA , $C'A'$; AD , $A'D'$; BD , $B'D'$, cut in five points of a straight s , the remaining sides CD and $C'D'$ cut also upon s .

In virtue of the hypothesis the triangles, ABC , $A'B'C'$, are in perspective, whence (art. xii.) the straights, AA' , BB' , CC' , concur in the same point O . For the same reason the triangles ABD , $A'B'D'$ are in perspective, wherefore AA' , BB' , DD' concur in a point, but we have already seen that AA' , BB' concur in O , wherefore DD' likewise passes through O , whence the

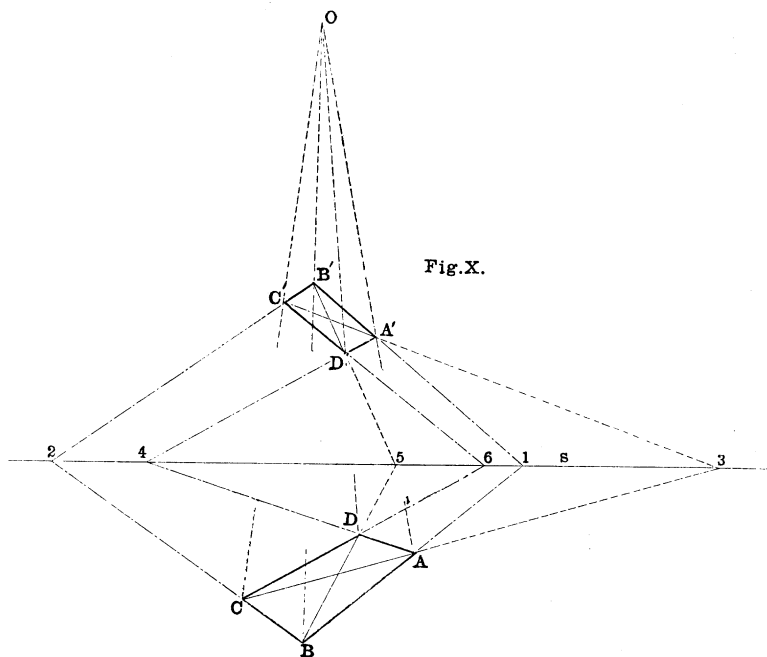
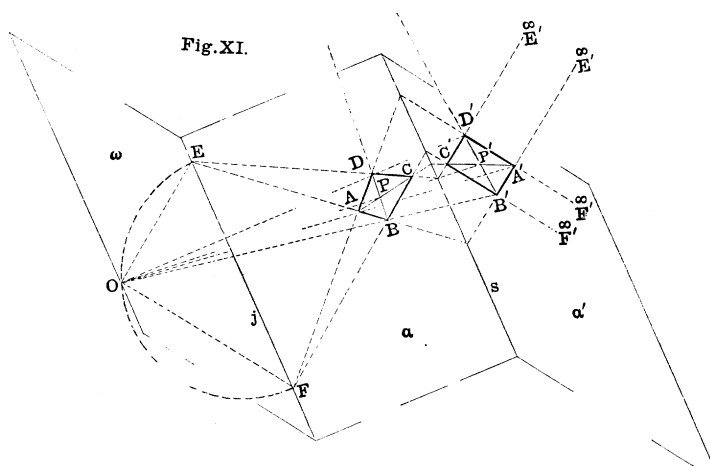


Fig. X.

triangles BCD , $B'C'D'$ are also in perspective, for BB' , CC' , DD' concur in the point O , when CD , $C'D'$ must meet on the line s .

XXII. LAMBERT'S Problem. *Projection of an Irregular Quadrilateral into a Rectangle.*—That a pencil of rays converging to a point may be projected into a pencil of parallel rays, *i.e.* into a pencil of rays converging to a point at infinity (xvi.), enables us to project an irregular quadrilateral into a rectangle.

On the plane α (fig. xi.) let $ABCD$ be any irregular quadrilateral and choose for the line j that line which goes through the intersections EF of the two pairs of opposing sides, AB , CD and AD , BC . Upon the plane ω , parallel to α' , and upon EF , as hypotenuse, construct the right-angled triangle EOF , then O is the centre of projection, and the rays converging in E on the plane α are (on the plane α') parallel to OE (xvi.). Similarly, the rays converging in F on α are (on the plane α') parallel to OF , whence $ABCD$ has been projected on α' into a rectangle $A'B'C'D'$.

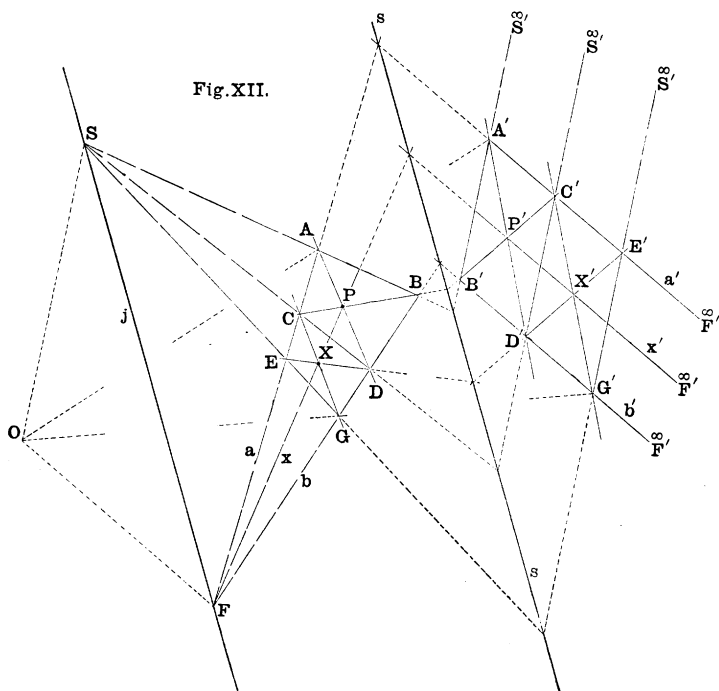


This projection holds true when ω and a' are turned round parallel to each other till they are in the same plane with a , and therefore holds true with our figure, the plane a being the plane of the paper (Remark art. xva.). It is often convenient, however, as already remarked, to conceive the figure as being upon the several planes ω , a , a' .

XXIII. *In Projecting the Irregular Quadrilateral into a Rectangle, the Locus of the Centre of Projection is the Surface of a Sphere.*—Let now a circle be described on EF as a diameter. If O is located at any point on that circle, the projection on a' will be a rectangle, and if ω and a' are turned round j and s this circle will describe a sphere, whence the centre O of projection may have its locus on any point on this sphere, projecting the given rectangle into a quadrilateral.

XXIV. *Given Two Rays directed to an Inaccessible Point F , to find another Ray directed to that Point and Passing Through a Given Point.*—Art. xxii. furnishes us with the means of solving the above. Given two lines a and b (fig. xii.) converging to an inaccessible point F . It is required to draw through a given point P a line x which shall also go through F .

Draw any line AB cutting the two lines a and b in A and B .



Through A and B draw AP , BP cutting a and b in D and C . Join CD cutting AB in S . Through S draw any other line SG cutting a and b in E and G , draw the interior diagonals of the quadrilateral $ECDG$ intersecting in some point X . Through P and X draw the line x . a , b and x will cut in the same point F .

For, project the quadrilateral $ABGE$ so as to form a parallelogram. It will also necessarily form two parallelograms, and the projections of P and X will be the bisections of the two pairs of diagonals, and the line x' drawn through them will therefore be parallel to $A'E'$, $B'G'$, and will consequently meet in the infinitely distant point F' , whence the corresponding line x goes through F .

XXVa.—*The Centre of Homology its own Correspondent.*—Conceive fig. vi. to be the orthographic projection on the plane of the paper of a pyramid $OA'B'C'$ cut by two planes a and a'

we readily perceive that this orthographic projection may be so taken, as to reduce some one ray or arête of the pyramid as OAA' to a point, that is, the orthographic plane might be so chosen as to be perpendicular to that arête, which would necessarily become represented by the point O , O is therefore its own correspondent. In this disposition of two homologous figures where one of the arêtes of a cone is reduced to a point, the centre of homology, the tangent to the one figure at this point is likewise the tangent to the other at the same point. This disposition of the centre of homology gives rise to a material simplification of propositions in conics as we will afterwards find. See figures lv. and lvi.

XXVb. *The Axis of Homology its own correspondent.*—We readily perceive that if a curve on the plane a have s for its tangent, then the homologous curve on the plane a' has likewise s for its tangent through the same point.

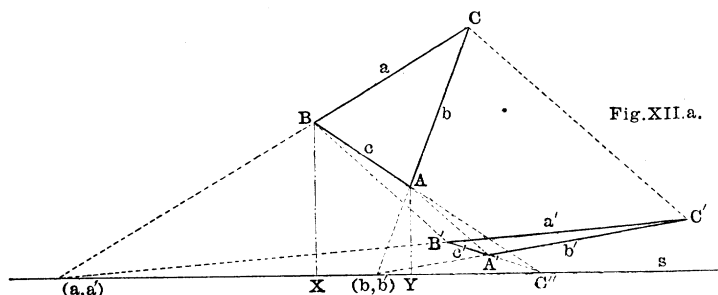
XXVIa. *The Centre of Homology O is the Centre of Similitude when the Axis s of Homology is at Infinity*, for then corresponding straights are always parallel. O is in this case said to be the centre of similitude, and corresponding figures are similar and similarly placed.

Polygons of an even degree, whose several pairs of opposite sides are equal and parallel, of which two similar parallelograms are at the one end of the class and two circles at the other end, have two centres of similitude, external and internal.¹

XXVIb. *The Centre of Homology O may Itself be at Infinity*, then straights joining corresponding points in two figures are parallel to a fixed direction, and such figures are said to be *homologous by affinity*; thus an ellipse corresponds to an ellipse, a hyperbola to a hyperbola, a parabola to a parabola, a parallelogram to a parallelogram.

XXVIc. *Theorem.*—In two triangles, ABC , $A'B'C'$, fig. xii. a , homologous by affinity, *i.e.* having their centre of projection O at infinity, the parallel lines joining the three pairs of summits are proportional to the perpendiculars let fall from either set of

¹ Townsend's *Modern Geom.* vol. i. pp. 35-40.



summits upon the axis of homology. Let the point (cc') on s be named C'' and the perpendiculars from A and B upon s be AA' , BB' , then from similar triangles we have the following proportions

$$\begin{aligned} C''A : C''B &:: C''A' : C''B' \\ &:: AA' : BB' \\ &:: AY : BX. \end{aligned}$$

Problem.—The above gives the solution of the following problem. Given three points, A , B , C , and three distances, p , q , r , as AA' , BB' , CC' , respectively proportional to the perpendiculars from a given line, to describe that line. Through A , B , and C draw p , q , and r parallel to each other, their extremities are the summits of another triangle, $A'B'C'$, homologous by affinity. Construct the line s of these two homologous triangles, it is the line required. This problem is employed in arts. 213 and 215, pp. 277 and 279, in treating of the *Elastic Arch*.

XXVI*d*. The Parabola or Figure Homologous to the Circle with One Point at Infinity.—This will be recognised in fig. xxix. of this chapter, where the point J in the circle corresponds to the point J' at infinity in the homologous figure, SJ giving the direction of the point at infinity. SJ is an arête of the cone vertex s , the circle a section of the cone on plane α , the parabola a section on the plane α' parallel to the arête SJ .

XXVI*e*. The Hyperbola or Figure Homologous to the Circle with Two Points at Infinity.—This will be recognised in figs. 165 and 167, pp. 305 and 308, Chap. VII, where the points M and N in the circle correspond to the points at infinity in the

homologous figure, and SM , SN are the directions of the tangents to the points at infinity, *i.e.* of the asymptotes. The projection of that part of the circle to the left of the line j necessarily approaches the axis s from infinity, from left to right, and forms the other branch of the hyperbola.

Remark.—In all our homologous figures we have kept to one arrangement of O or S , j , and s , so as not to confuse the student in his first conceptions of the principle, with the exception of fig. xxv., and in the section, fig. lv., on forms projective in the conics we will employ the arrangement indicated in xxv. *a*, xxv. *b*. Fig. lvi. is likewise an arrangement deviating from the normal form.

Section III.—Harmonic Ratio.

XXVII. *Any One Diagonal of a Complete Quadrilateral is divided Harmonically by the Other Two Diagonals.*—For fig. xiii. project the quadrilateral $ABCD$ into a parallelogram $A'B'C'D'$.

(1) The diagonal AC is divided harmonically by the other two diagonals DB , EF in the points P and G , *i.e.*

$$\frac{AG}{CG} = -\frac{AP}{CP} \text{ (art. ix.)}$$

For, in the plane $OCPAG$, we have the rays from O , c , p , a , g , cut by two transversals, $CPAG$ and $C'P'A'G'$, in which G' is at infinity, whence the double, or anharmonic ratio becomes

$$\frac{AG}{CG} : \frac{AP}{CP} = \frac{A'P'}{C'P'} = -1,$$

or

$$\frac{AG}{CG} = -\frac{AP}{CP}.$$

(2) Similar reasoning in reference to the diagonal DB would give

$$\frac{BH}{DH} : \frac{BP}{DP} = \frac{B'P'}{D'P'} = -1,$$

or

$$\frac{BH}{DH} = -\frac{BP}{DP}.$$

(3) The projection of the third or exterior diagonal EF is wholly at infinity, but we obtain the double ratio thus. Take C for the centre of projection of the rays CE , CF , CG , CH , then $HDPB$ being the points in a transversal, which has been proved above to be harmonic, and $HEGF$ are the corresponding points in another transversal of the same pencil of rays, whence

$$\frac{FH}{EH} = -\frac{FG}{EG}.$$

It will be instructive to demonstrate this last ratio by means of the relation established among the sines in art. vi. For, supposing the projection upon the plane α to be a rectangle, (for (vii.) this does not alter the anharmonic ratio,) then we have the following

$$\frac{\sin \overset{\infty}{FP'}\overset{\infty}{H}}{\sin \overset{\infty}{EP'}\overset{\infty}{H}} = -\frac{\sin \overset{\infty}{FP'}\overset{\infty}{G}}{\sin \overset{\infty}{EP'}\overset{\infty}{G}}.$$

For, let $\angle A'P'D' = a$ then $\angle A'P'B' = 180^\circ - a$, and the above expression becomes

$$\frac{\sin \{\frac{1}{2}(180^\circ - a) + a\}}{\sin \frac{1}{2} a} = -\frac{\sin \frac{1}{2}(180^\circ - a)}{\sin \frac{1}{2} a},$$

for

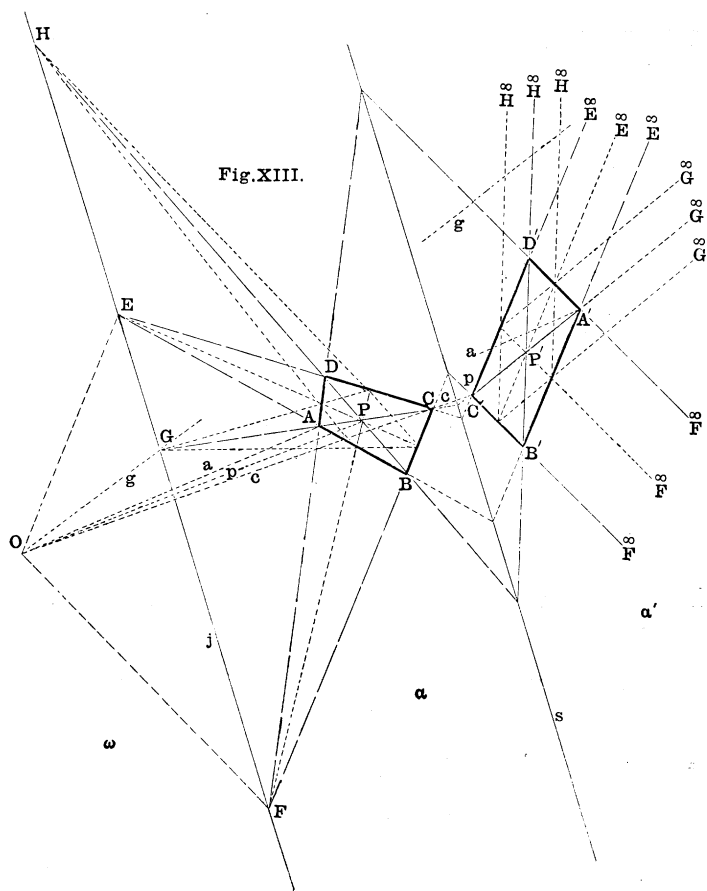
$$\sin \{\frac{1}{2}(180^\circ - a) + a\} = \sin (90^\circ + \frac{1}{2}a)$$

and

$$\sin \frac{1}{2}(180^\circ - a) = \sin (90^\circ - \frac{1}{2}a),$$

of which the left-hand members are equal in value.

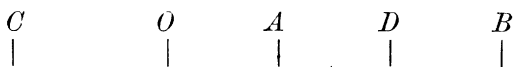
Again, if through the intersection P of the diagonals of the parallelogram, fig. xiii., we draw lines parallel to its sides, we obtain the diagonals of a new parallelogram inscribed to the first, such that ^{diagonals} _{sides} of one parallelogram are parallel to ^{sides} _{diagonals} of the other, or, otherwise, the diagonal of the one (as BD) meets the two sides parallel to it of the other in a point (as $\overset{\infty}{H}$) at infinity. Whence it follows, with regard to



their projection on the plane a into circumscribed and inscribed quadrilaterals, the diagonals of both intersect in the point P , and a diagonal of circumscribing quadrilateral, meet in a point inscribing quadrilateral, on that line j , the two sides of the inscribing quadrilateral which lie on either side of that diagonal.

This is evidently involved in the harmonic properties of the first considered, or circumscribing quadrilateral.

XXVIII. *Centre O of Segment CD dividing a Given Straight AB in Harmonic Ratio.*—Let the line $ABCD$ be supposed divided harmonically



then

$$(ABCD) = -1,$$

or

$$\frac{AC}{BC} : \frac{AD}{BD} = -1,$$

or

$$\frac{AC}{BC} + \frac{AD}{BD} = 0 \quad . \quad . \quad . \quad (1),$$

or

$$\frac{AC}{AD} + \frac{BC}{BD} = 0 \quad . \quad . \quad . \quad (2).$$

Let O be the middle of the segment CD or $OD = CO = -OC$, we have

$$AC = OC - OA, \quad AD = OD - OA = -(OC + OA)$$

$$BC = OC - OB, \quad BD = -(OC + OB),$$

whence, substituting in (2)

$$\frac{OC - OA}{OC + OA} = \frac{OB - OC}{OB + OC},$$

or

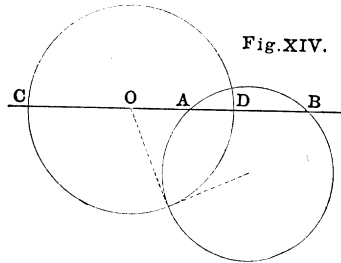
$$\frac{OC}{OA} = \frac{OB}{OC},$$

wherefore

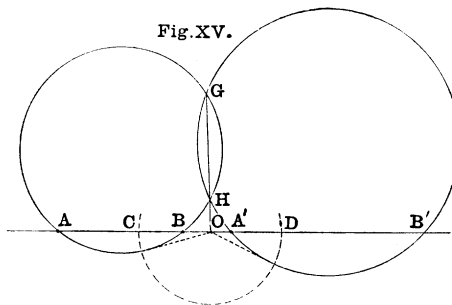
$$OC^2 = OA \cdot OB \quad . \quad (3).$$

This last (3) equation shows that OA , OB have the same sign, that is O never falls between A and B , and therefore the half of the segment CD is a mean proportional between the distance of the points A and B from the central point of that segment CD .

For reasons which will afterwards appear, this point O is called the centre of involution and D the focus of involution, $OA \cdot OB$ the constant rectangle of involution.



XXIX. If a Circle is described through the Points A and B , OC will be the Length of its Tangent from O (fig. xiv.); wherefore the Circle with OC as Radius will cut the First Circle (that through AB) Orthogonally. — Reciprocally. If two circles cut each other at right angles they will cut any straight passing through one of their centres in four points harmonically separated from each other. These are simply corollaries from last article, xxviii.

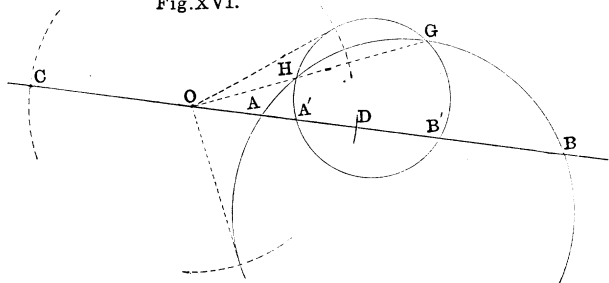


Being given in a straight, figs. xv. and xvi., two pairs of points AB , $A'B'$ to determine another pair CD such that the two groups $ABCD$, $A'B'CD$ may be harmonically separated.

Take any point G not in the straight and describe the circles GAB , $GA'B'$ and let H be their second point of intersection, draw GH and let O be the intersection of that straight with the given straight, we will have in the first circle

$$OA \cdot OB = OG \cdot OH,$$

Fig. XVI.



in the second circle

$$OA' \cdot OB' = OG \cdot OH,$$

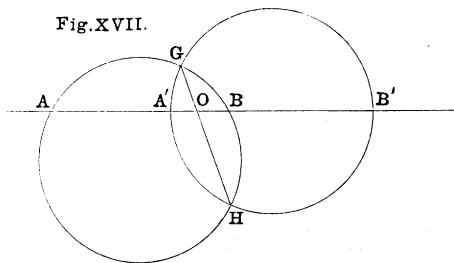
consequently

$$OA \cdot OB = OA' \cdot OB' = OC^2,$$

wherefore O is the middle point of the segment required, and taking the length OC or OD of the tangent from O as radius, describe from O a circle cutting AB and $A'B'$ in the points C and D , C and D are the points required.

XXX. *Real and Imaginary Solutions of the Foregoing Problem.*—The above problem admits of a real solution when the point O falls exteriorly to the two segments AB and $A'B'$, and consequently exterior to the two circles (figs. xv., xvi.).

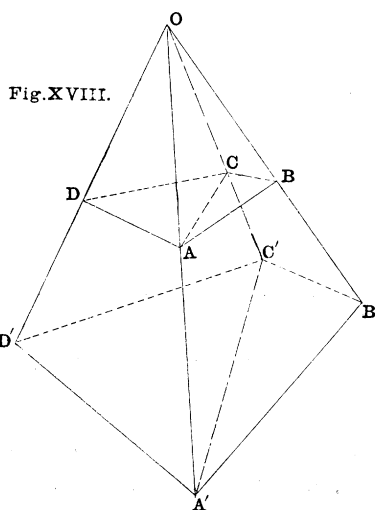
Fig. XVII.



There exists no real solution when the points AB are separated by the pair $A'B'$ (fig. xvii.). In this case the point O falls within both segments.

XXXI. *Double Point.*—Let $ABCD$ be four points separated harmonically and let A and B be infinitely near, *i.e.* coincident. D is a double point. Further, if C is at an infinite distance, D

will coincide with A and B and will become the middle point O of these two points A and B .



Section IV.—Poles and Polars in Conic Sections.

XXXII. PONCELET'S *Condition of Projectivity*.—This is an extension of art. vi. Consider the pyramidal figure (fig. xviii). Let $A, B, C, D \dots$ be the different points of the figure and O the centre of projection, and from O let rays pass through $A, B, C, D \dots$. Upon these rays let other points $A', B', C', D' \dots$ be taken. Join these points among themselves $AB, BC, CD, BD \dots A'B', B'C', C'D', B'D' \dots$. The figure $A'B'C' \dots$ is a projection of a kind of the figure $ABC \dots$.

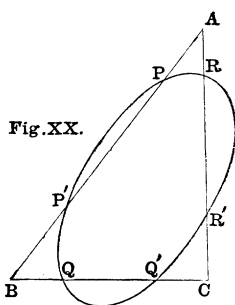
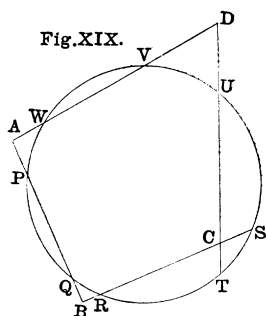
Let us now have any equation between the lines $AB, BC, CD \dots$ but such as involves the same points in each member of it, such as

$$AB \cdot CD + k \cdot BC \cdot DA + l \cdot AC \cdot BD = 0,$$

in which, substituting the values in terms of the lengths of the rays and sines of the angles, we obtain (art. vi.)

$$abcd \sin a'b \sin c'd + k \cdot bcda \sin b'c \sin d'a \\ + l \cdot acbd \sin a'c \sin b'd = 0.$$

We can divide every term of this expression by $abcd$, when there remains only a relation among the sines.



XXXIII. CARNOT'S *Theorem*.—Consider the intersections of a conic by a given polygon (fig. xix.) A, B, C, D, \dots its summits PQ, RS, TU, \dots the points in which it cuts the conic, then Carnot's theorem is

$$\begin{aligned} AP \cdot AQ \times BR \cdot BS \times CU \cdot CT \dots \\ = AW \cdot AV \times BQ \cdot BP \times CR \cdot CS \dots \quad (1) \end{aligned}$$

We can verify the truth of this expression in the case of the circle for (*Euclid* iii., 36)

$$\begin{aligned} AP \cdot AQ = AW \cdot AV, \quad BR \cdot BS = BQ \cdot BP, \\ CU \cdot CT = CR \cdot CS \dots \dots \quad (2) \end{aligned}$$

whence the truth of the expression in this case is evident.

Now this expression is projective, for each point occurs the same number of times on either side of the equation, whence it is true for any conic section whatever.

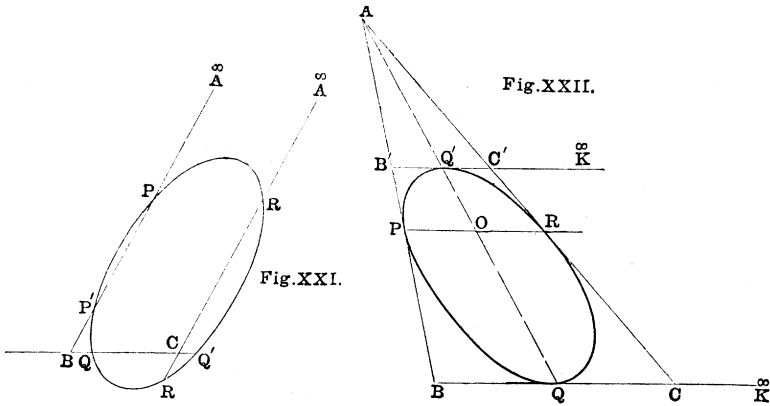
Reducing the number of the sides of our polygon to three (fig. xx.) we have

$$\begin{aligned} AP \cdot AP' \times BQ \cdot BQ' \times CR \cdot CR' \\ = BP \cdot BP' \times CQ \cdot CQ' \times AR \cdot AR' \dots \quad (3) \end{aligned}$$

XXXIV. *Removal of Points to Infinity and other Simplifications in Carnot's Theorem.*

i. Suppose, further, the point A (fig. xxi.) removed to infinity, the expression becomes

$$BQ \cdot BQ' \times CR \cdot CR' = BP \cdot BP' \times CQ \cdot CQ' \dots \quad (4)$$



or

$$\frac{BP \cdot BP'}{BQ \cdot BQ'} = \frac{CR \cdot CR'}{CQ \cdot CQ'} \quad \dots \dots \dots (5)$$

that is, in words: The product of the ordinates $BP \cdot BP'$ has to that of the corresponding abscissa $BQ \cdot BQ'$ a *constant ratio* for all parallels to AB (and which would subsist were the curve replaced by any two right lines).

ii. Suppose, again, that the triangle circumscribe the conic (fig. xxii.), then the points P and P' , Q and Q' , R and R' coincide, and the expression (3) becomes

$$AP^2 \cdot BQ^2 \cdot CR^2 = BP^2 \cdot CQ^2 \cdot AR^2 \quad \dots \dots (6)$$

extracting $\sqrt{}$

$$AP \cdot BQ \cdot CR = BP \cdot CQ \cdot AR. \quad \dots \dots (7)$$

iii. Let, further, the side BC of the circumscribed triangle be led parallel to the cord of contact PR , opposite the angle A , then we have

$$\frac{AP}{BP} = \frac{AR}{CR} \quad \dots \dots \dots (8)$$

wherefore

$$BQ = CQ \quad \dots \dots \dots (9)$$

and consequently the point Q belongs to the straight OA which passes through the summit of the angle A , and through the middle O of the cord of contact PR .

XXXV. *Diameter of a Conic Section.*—In leading the new tangent $B'C'$ parallel to the tangent BC (fig. xxii.) we can conclude in the same manner that its point of contact Q' is upon the straight AO , and thus this straight appertains at the same time to all the chords of conic sections parallel to PR and to the tangents BC , $B'C'$, *i.e.* the parallel chords of conic sections have their middle points, and the tangents parallel to these chords have their tangent at points distributed on the same straight called the diameter.

XXXVI. *Harmonic Division of the Diameter of a Conic Section.*—In the same manner, if we observe that the three tangents, BQ , BB' , $B'Q'$ may be considered to form a circumscribed triangle $BB'\check{K}$ of which the sides $B\check{K}$, $B'\check{K}$ meet at infinity we can deduce

$$PB \cdot Q'B' \cdot \check{K}Q = PB' \cdot QB \cdot Q'\check{K} \quad . \quad . \quad . \quad (7a)$$

$$PB \cdot Q'B' = PB' \cdot QB \quad . \quad . \quad . \quad . \quad (7b)$$

$$\text{or} \quad \frac{PB}{PB'} = \frac{QB}{Q'B'}; \quad . \quad . \quad . \quad . \quad (8a)$$

but since BQ , OP , and $B'Q'$ are parallel

$$\frac{PB}{PB'} = \frac{OQ}{OQ'} \quad \text{and} \quad \frac{QB}{Q'B'} = \frac{AQ}{AQ'} \quad . \quad . \quad . \quad . \quad (10)$$

wherefore

$$\frac{OQ}{OQ'} = \frac{AQ}{AQ'} \quad \text{or} \quad \frac{OQ}{AQ} = \frac{OQ'}{AQ'} \quad . \quad . \quad . \quad . \quad (11)$$

that is, the diameter QQ' is divided harmonically by the chord PR and by the summit A of the circumscribing angle.

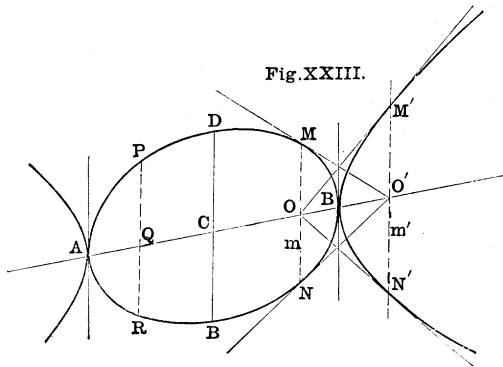
ii. When the conic section is a parabola, that is, when one extremity of the diameter is at infinity. This occurs when, for instance, the summit of the cone is as it were the point O on the plane ω , represented in fig. lvii. by S , a circular section of the cone so placed on the plane α that it touches the line j in a point J , then the section of the cone on the plane α' is a parabola, in which the correspondent J' of the point J where the circle touches j is at infinity, for the plane of section on α' is parallel to the arête SJ of the cone.

Let the extremity Q of the diameter (fig. xxii.) pass to infinity, then the curve becomes a parabola, and equation (11) becomes

$$\infty = \frac{OQ'}{AQ'},$$

that is,

$$OQ' = Q'A \dots \dots \dots (12)$$



XXXVII. *The Relation between the Square of a Semichord, Real or Ideal, and the Rectangle formed by the Parts into which it divides its Corresponding Diameter.*

i. Let MN and PR (fig. xxiii) be two parallel chords of any conic section, AB the diameter conjugate to their common direction, that is, that which passes through their middle points O and Q (art. xxxv.), the expression (5) becomes

$$\frac{OM^2}{OA \cdot OB} = \frac{PQ^2}{QA \cdot QB} = \frac{O'M'^2}{O'A \cdot O'B} = p \dots \dots (13)$$

whence

$$\left. \begin{aligned} OM^2 &= p \cdot OA \cdot OB \\ O'M'^2 &= p \cdot O'A \cdot O'B \end{aligned} \right\} \dots \dots \dots (14)$$

p being a constant quantity, which varies only with the direction of the diameter AB , and represents, in the case of the ellipse and of the hyperbola, the inverse ratio of the square of this diameter to its conjugate.

It will be seen that $O'M'$ is an ideal semi-chord beyond the limits of the conic.

ii. *Let the Conic be a Parabola.*—Then one of the extremities of the diameter AB as B passes to infinity, and expression (13) becomes

$$\frac{OM^2}{OA} = \frac{PQ^2}{QA} = p \quad \dots \quad (15)$$

whence

$$OM^2 = p \cdot OA.$$

We give here several Theorems without formal demonstration.

XXXVIII. *The Diametral Plane in a Cone.*—The centres of all parallel chords of a cone surface are included in a plane called diametral, and which includes the summit of the cone.

XXXIX. *Ideal Chord common to different Conic Sections.*—Let S (fig. xxiv.) be the vertex of a cone, and m a straight. Conceive the diametral plane conjugate to the direction of m , dividing the chords which are parallel to m into two equal parts it will cut the cone along the two arêtes SA, SB , and the two sectional planes $AO, A'O$ in this plane cutting in the point O , the line m of which the parts $AB, A'B'$ terminated by the arêtes SA, SB will be the diameters of the two sections conjugate to the direction of m . This is the first condition that m be an ideal chord.

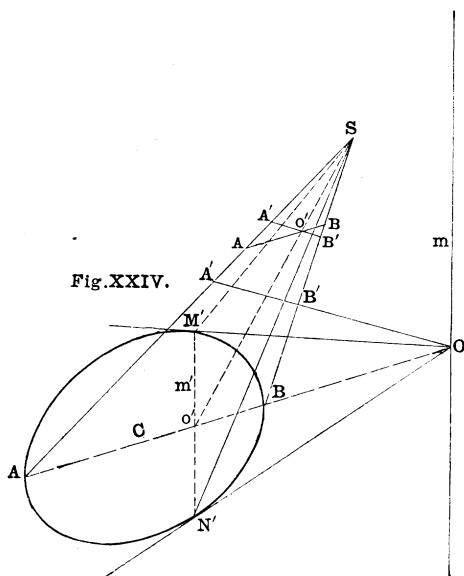
It is necessary that the constants p and p' appertaining to the diameters AB and $A'B'$ respectively, satisfy the equation

$$p \cdot OA \cdot OB = p'OA' \cdot OB' = OM^2 \quad \dots \quad (14a)$$

for m (i.e. $2 \cdot OM$) is their common ideal chord.

For we can always determine upon the surface of the cone two sections $AB, A'B'$, parallel, and consequently similar to the first two, and which have a real chord in common, and in which therefore the above relations between the lines and the constants have place, and which are necessarily equal to the constants p and p' of the conic sections in question AB and $A'B'$.

XL. *Ideal Conic Sections through the Summit of a Cone.*—Suppose now that the plane AB be turned round m as on a hinge until it passes through the summit S of the cone, we may



regard its intersection there as a conic section, infinitely small and similar to that which a parallel plane would give, and having consequently the same constant p'' in reference to the direction SO , and m (MN , letters not given in figure) being the ideal chord, we have

$$OM^2 = p \cdot OA \cdot OB = p' \cdot OA' \cdot OB' = p'' \cdot OS \cdot OS = p'' \cdot OS^2,$$

that is, the ratio of OM to OS represents the ratio of the conjugate diameters of a conic section parallel to the plane Sm .

XLI. Ideal Circular Conic Section through the Summit of a Cone.—Suppose now that the plane Sm be parallel to a circular section of the cone, SO must evidently be perpendicular to m , and $p'' = \text{unity}$, whence

$$OM = OS.$$

XLII. Projection of a Conic Section into a Circle.—Being given a conic section C and a straight i' (fig. xxv.) on the same plane a' , to find a centre of projection as S and a plane as a such that

the straight i' will be projected upon it to infinity, and the conic section at the same time be projected into a circle.

Let S be the unknown centre of projection, then the conditions of the problem require that the plane ω' which contains S and i' be parallel to the plane α of projection, and that this plane α must cut the cone of which C is the base and S the vertex in a circle.

In order to fulfil these conditions, we must have first of all, the straight i' exterior to the conic sections. Let the plane of the conic section C turn round i' as a hinge until it pass through the vertex S of the cone, while the tangents led from O' touch the cone following the arêtes MS , NS . We can regard it as cutting the cone at S in a section infinitely small and similar to a conic in a plane parallel to itself (Si') and having consequently the same constant p' in reference to SO' which now represents the direction of the conjugate diameter to the ideal chord

$$OM^2 = p \cdot OA \cdot OB = p' \cdot O'S^2.$$

Suppose now that the plane Si' be parallel to the plane α of a circular section of the cone, SO' will evidently be perpendicular to i' , and $p' = \text{unity}$, so that $O'M' = O'S$. Wherefore the locus of the vertex S of a cone lies in a plane through O' , perpendicular to m and in a circle upon that plane whose centre is O' and whose radius $O'S$ is equal to $O'M'$.

In the plane α parallel to the plane Si' or ω' in which $O'M' = O'S$, the line i' is projected to infinity, for ω' and α are parallel, $O'BOA$ are points in a transversal of a pencil of rays, SO' , SB , SO , SA and $\overset{\circ}{O}B'C''A''$ are points in another transversal of the same pencil, wherein $\overset{\circ}{O}$ is at infinity, and the ratio of the first pencil being harmonic that of the other is necessarily so, or

$$\frac{O'A}{O'B} = - \frac{OA}{OB}$$

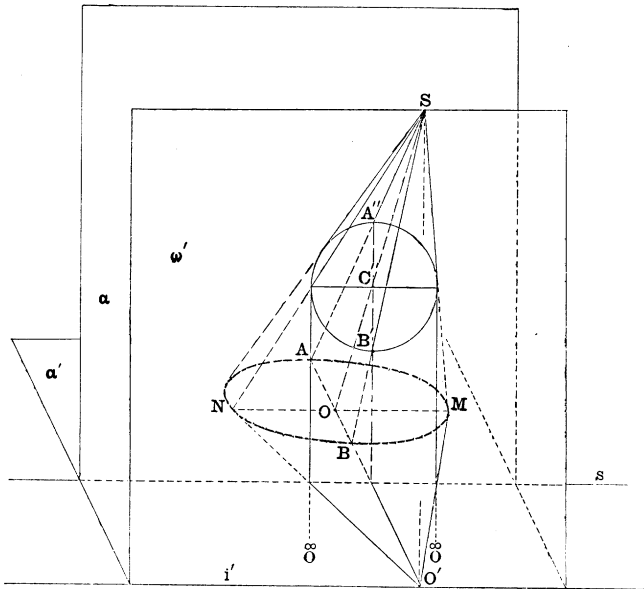
whence

$$\frac{\overset{\circ}{O}A''}{\overset{\circ}{O}B'} = - \frac{C'A''}{C'B'}$$

and

$$C''A'' = C'B'.$$

Fig. XXV.



XLIII. *Definitions of Pole and Polar.*—The point O (fig. xxiv.) is said to be the pole of the straight m' in reference to the conic C ; and the straight m' is said to be the polar of the point O . Also the point O' is said to be the pole of the straight m ; and m is said to be the polar of the point O' .

XLIV. *Projection of a Conic Section, with Inscribed and Circumscribed Quadrilaterals and Deductions relating to Poles and Polars therefrom.*—Any quadrilateral being inscribed to a conic, we may regard the figure as the projection of another, for which the exterior diagonal is projected to infinity, and the conic becomes a circle, whilst the inscribed quadrilateral is transformed into a rectangle.

For, let $ABCD$ (fig. xxvi.) be the rectangle in question, all the properties which appertain to it and to the circle, appertain to the primitive figure. Through each of the summits A, B, C, D lead a tangent to the circle in order to form the circumscribed parallelogram $\check{A}\check{B}\check{C}\check{D}$, of which the opposite sides concur at infinity. Trace the diagonals AC and BD , $\check{A}\check{C}$ and $\check{B}\check{D}$, they will pass (as known from elementary geometry) through the

centre P , and the two last $\check{A}\check{C}$ and $\check{B}\check{D}$ will be parallel to the sides of the rectangle $ABCD$, *i.e.* will concur with them at infinity. Likewise, all straights going through the centre P and terminated at the circumference or at the opposite sides of the two parallelograms are divided into equal parts by this point P , and are thus cut harmonically by P and the point at infinity. Turn now to the primitive figure. The straights, which among themselves are parallel in the projection, will in the primitive concur in a point on the line j , which in the projection is at infinity. The intersection P of the interior diagonals of the quadrilateral will be the pole of this straight. Whence, bearing in mind the preceding propositions, we conclude that

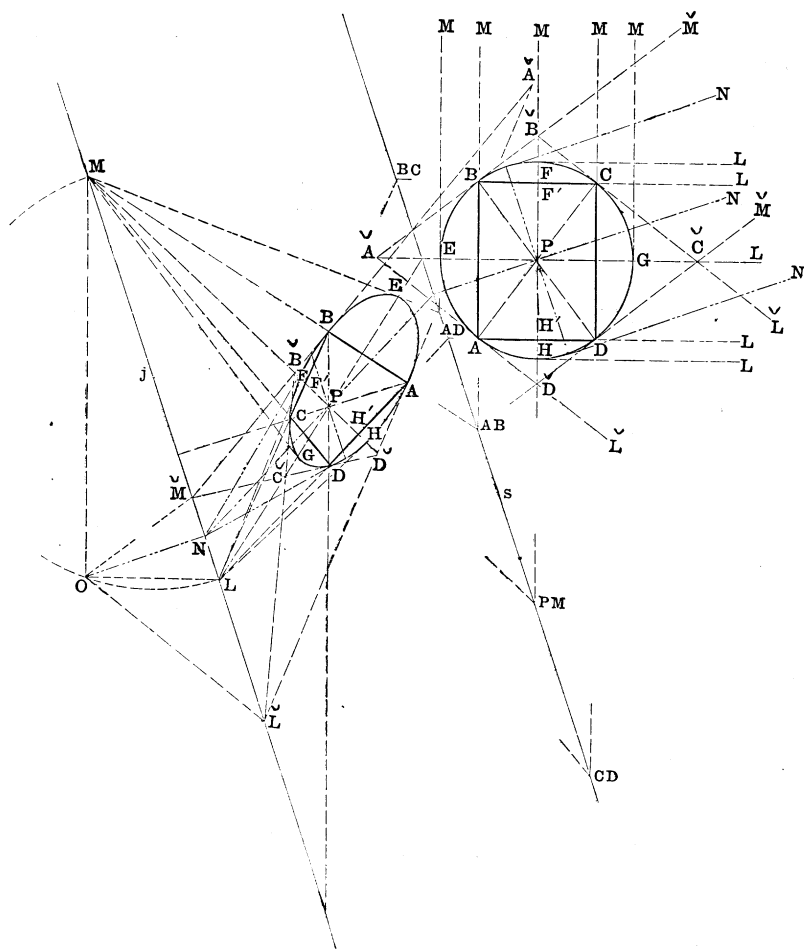
If we inscribe to a conic section any quadrilateral $ABCD$ and to the same conic circumscribe another quadrilateral $\check{A}\check{B}\check{C}\check{D}$, of which the sides touch the curve at the summits of the other—

1. The four interior diagonals will cross in the same point P .
2. The points of concurrence L and M , \check{L} and \check{M} of the opposite sides of the two quadrilaterals and of the tangents from $\frac{L}{M}$ upon the intersections of the conic with $\frac{MP}{LP}$ will be on the polar of P .
3. The diagonals of the circumscribed quadrilateral will likewise concur in the points L and M .
4. Each of these last points is the pole of the straight or diagonal which passes through the other, for it is the point of concurrence of the tangents which correspond to that diagonal.
5. All straights passing through the point P and terminated by the conic section, or by two of the opposite sides of one of the quadrilaterals is divided harmonically by that point, and by the point where it is intersected by its polar j .

And reciprocally—

6. If we inscribe in a conic section a series of chords AB , $A'B'$, $A''B''$ fig. xxvii., all directed towards a point P , chosen at will upon the plane of the curve, all C , C' , C'' which are, in reference to the chords, the fourth conjugate harmonics of the point P , are situated on the straight line polar to P .

Fig. XXVI.



7. The intersections L and M , L' and M' of the diagonals and opposite sides, and the points of concurrence T , T' of pairs of tangents from the extremities of each chord are upon the same straight as C , C' , C''

For, if we project the point P to infinity, whilst we at the same time project the conic into a circle (this may be done by projecting the point P and any inscribed quadrilateral

$UVWX$, two of whose sides as UV , WX concur in P , and UW , VX in a point Y , which is projected to infinity along with P into a rectangle), then (fig. xxvii.) AB , $A'B'$ become parallel chords, and LMT , $L'M'T'$ from elementary geometrical considerations, lie upon a diameter bisecting these chords and at right angles to them, and therefore divided by $C\bar{P}$, $C'\bar{P}$ harmonically.

Reciprocally, if from points T , T' taken in an arbitrary straight in the plane of the conic section, we lead pairs of tangents to that curve, their corresponding chords of contact AB , $A'B'$ will all concur in an unique point P , pole of that straight.

These properties may be thus enunciated:—

1. If a straight line in the plane of a conic section pivot around a fixed point, the pole of that straight will trace a straight line.

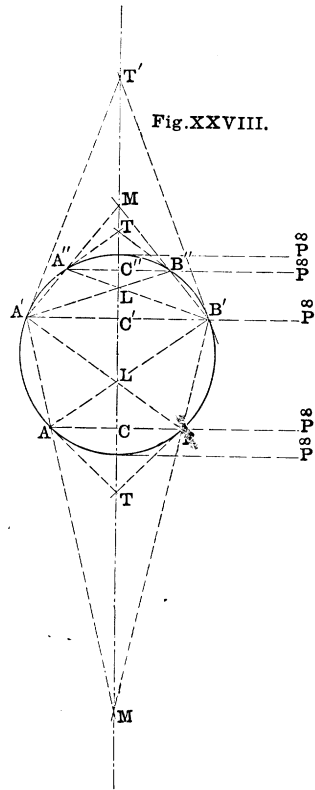
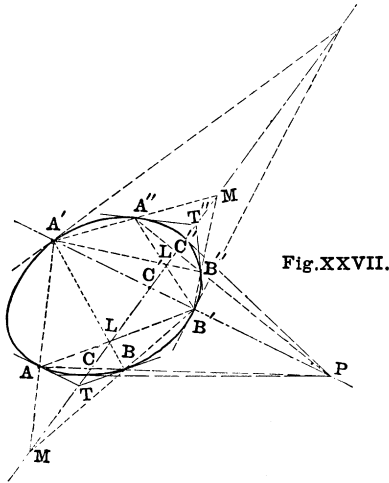
2. Reciprocally, If a point move along a straight line traced in the plane of a conic section, the polar of this point will pivot around a fixed point, the pole of the straight line.

In fig. xxvi. j is the polar of P , and a line through P parallel to j is polar to a point N on j , the foot of the perpendicular from O on j . For in the circular projection, the real chord and its projection being parallel meeting s in infinity, and the conjugate diameter in the projection being at right angles, and ON being parallel to it, must be at right angles to j . The ideal chord j is thus the polar of P , and the real chord through P is the polar of the point N , and lines from N to the extremities of the real chord are tangents to the conic.

XLV. Problems regarding Poles and Polars.

i. From a given point M (fig. xxvi.), exterior to a given conic, to lead to it two tangents. From M lead two arbitrary secants MBA , MCD , cutting the conic in $ABCD$. Complete the quadrilateral $ABCD$ by joining AC , BD , BC , AD , giving by their intersections the two points P and L . Join LP , it will cut the conic in G and E , MG , ME are the tangents required.

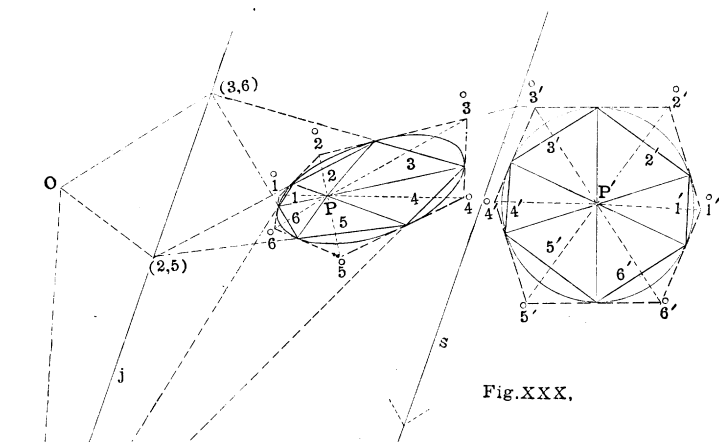
ii. Given the polar GE to find the pole M . Take any point P on GE , and through it draw CA , DB cutting the curve in A , B , C , D . AB and DC will intersect in the pole M .



iii. Given the polar j , to find the pole P . Take any point L on j , from which draw two secants LCB , LDA , cutting the curve in A, B, C, D . Join AC , BD , they intersect on the pole P of the line j .

iv. Given the pole P , to find the polar j . Through P draw AC , BD . Complete the quadrilateral, whose exterior diagonal ML is the polar required.

Pole and Polar in a Parabola.—In a parabola fig. xxix.,
i. That part of the straight PC going through the pole P in the direction of the point at infinity \tilde{J}' between the pole P and its polar AB is bisected by the curve. ii. The part of the polar cut off by the curve is bisected by this straight $P\tilde{J}'$. For, by



and 4', 2' and 5' in the projection are, two and two, parallel. The remaining sides 3' and 6' will likewise be parallel to each other. For the arcs subtended by 1' and 2' and by 4' and 5' are equal and opposite, whence the sides 3' and 6' subtend equal and opposite arcs and are thus equal and parallel. Wherefore in this projection we have three pairs of sides 1' and 4', 2' and 5', 3' and 6' meeting in three points on the line at infinity. Whence in the figure itself, they must concur on the line *j* in pairs in the three points (1, 4) (2, 5) (3, 6).

The line in which the three pairs of sides concur is usually called the Pascal line.

Perhaps it may be worth while to indicate mnemonically the concurring sides on the Pascal line. Writing the sides in their order in two rows, then their order in columns gives the concurring sides on the Pascal line. Thus

| | | |
|-----|-----|-----|
| 1 | 2 | 3 |
| and | and | and |
| 4 | 5 | 6 |

The points of the hexagon joined two and two
 summits (1, 2) (2, 3) (3, 4)
 and and and
 (4, 5) (5, 6) (6, 1)

concur in a point P . Of this we will not add a formal demonstration.

XLVII. BRIANCHON'S Theorem.—*In all Hexagons circumscribed to a Conic, the Diagonals which, two and two, join the opposite Summits intersect all three in the same Point, viz. the Pole of the Pascal Line.*—To the circle in fig. xxx., and through the summits of the previously inscribed hexagon, circumscribe a hexagon $\overset{\circ}{1}'$, $\overset{\circ}{2}'$ $\overset{\circ}{6}'$. This, in its projection upon the previously projected conic, will necessarily be a circumscribed hexagon to the conic, having its sides passing through the summits of the previously inscribed hexagon. Then we perceive from elementary considerations that in the circle, the diagonals

$$\begin{array}{ccc} \overset{\circ}{1}' & \overset{\circ}{2}' & \overset{\circ}{3}' \\ \text{and} & \text{and} & \text{and} \\ \overset{\circ}{4}' & \overset{\circ}{5}' & \overset{\circ}{6}' \end{array}$$

joining the opposite summits meet in the same point P as the diagonal of the inscribed hexagon.

This point P is the polar of the Pascal line. For any two sides of the hexagon as 2 and 5, form two sides of a quadrilateral whose opposite sides meet on j , and P is the intersection of its interior diagonals; 2 and 5 meet on j , and the other two sides (2, 3) (4, 5) and (1, 2) (6, 5) meet likewise on j , for in the projection their corresponding lines meet in an infinitely distant point on the line at infinity.

XLVIII. Constructions based upon PASCAL'S and BRIANCHON'S Theorems.

a. MACLAURIN'S Theorem.—*Being given Five Points in a Conic to describe the Conic, i.e. to find as many Sixth Points as we will.*—Let $ABCDE$ (fig. xxxi.) be five points in a conic; AB , BC , CD , DE , EX , XA , corresponding to sides 1, 2, 3, 4, 5, 6 of art. xlvi., then

AB , DE concur in the point (1, 4)

BC , EX " " " (2, 5)

CD , AX " " " (3, 6).

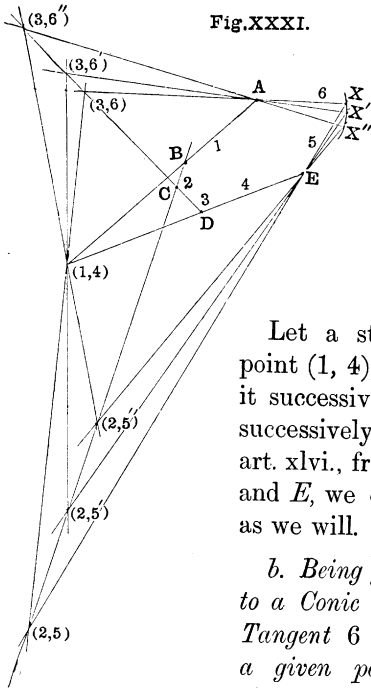


Fig. XXXI.

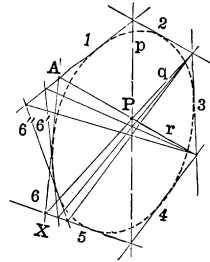


Fig. XXXII.

Let a straight line pivot around the point (1, 4), the couples of points in which it successively cuts the lines 2 and 3 are successively the points (2, 5) and (3, 6) of art. xlv., from whence, drawing lines to *A* and *E*, we obtain as many sixth points *X* as we will.

b. Being given Five Tangents 1, 2, 3, 4, 5 to a Conic (fig. xxxii.), to construct a Sixth Tangent 6 to the Curve, passing through a given point A in the first 1.—The diagonals are

$$\begin{array}{ccc} (1, 2) & (2, 3) & (3, 4) \\ \text{and} & \text{and} & \text{and} \\ (4, 5) & (5, 6) & (6, 1) \end{array}$$

or, concisely

$$p \qquad q \qquad r$$

Draw the diagonal *p*; lead the diagonal *r* through the points (3, 4) and *A*, cutting *p* in *P*; lead the diagonal *q* through the points (2, 3) and *P*, cutting the tangent 5 in *X*. Join *AX*, *AX* is the sixth tangent. Repeat thus for as many points *A*, *A'*, *A''*, as we will.

Formulating Notation of MACLAURIN'S Theorem mnemonically.

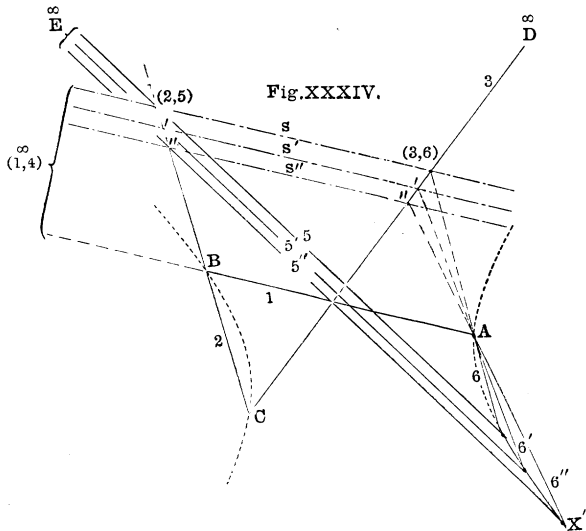
Points given *A* *B* *C* *D* *E* req^d. *X*,
that is (6, 1) (1, 2) (2, 3) (3, 4) (4, 5) „ (5, 6),
sides given 1 2 3 4 req^d. 5 and 6.

The Pascal line pivots round point $(1, 4)$, cutting sides 2 and 3 in points $(2, 5)$ $(2, 5')$ $(3, 6)$ $(3, 6')$, draw 5 parallel to 4 and 6 through $(6, 1)$ and $(3, 6)$.

In the problem thus conditioned, 4 and 5 have the direction of an asymptote, for they meet it in the point E at infinity.

b. Two of the Points may be at Infinity, fig. xxxiv., for example D and E , and the Problem may be thus stated :

| | | | | | | | |
|--------------|----------|----------|----------|-----------------------|-----------------------|--------------------|------------|
| Points given | A | B | C | $\overset{\infty}{D}$ | $\overset{\infty}{E}$ | req ^d . | X , |
| or | $(6, 1)$ | $(1, 2)$ | $(2, 3)$ | $(3, 4)$ | $(4, 5)$ | „ | $(5, 6)$, |
| sides given | | 1 | 2 | 3 | $\overset{\infty}{4}$ | req ^d . | 5 and 6. |

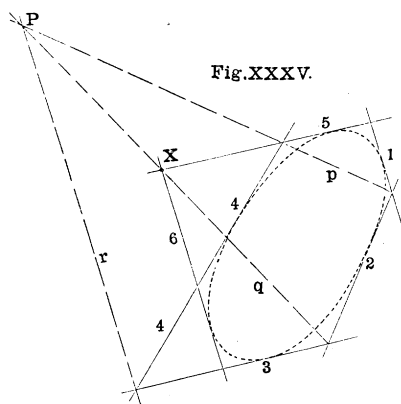


Here the side $\overset{\infty}{4}$ is wholly at infinity, its extremities being $\overset{\infty}{D}$ and $\overset{\infty}{E}$, 3 proceeds from its extremity $\overset{\infty}{D}$, 5 proceeds from its extremity $\overset{\infty}{E}$, whence 3 and 4 give the directions of the asymptotes. We perceive that in the Pascal line the point $(1, \overset{\infty}{4})$ is at infinity. This amounts to saying that the Pascal line pivoting round $(1, \overset{\infty}{4})$ remains parallel to 1, that the side 5 pivoting round E remains parallel to itself, whence the Pascal line moving parallel to 1 cuts 3 in a series of points $(3, 6)$, and 2 in a series

of points (2, 5), through which we draw 5 parallel to itself, whence we may obtain a series of points X . This problem may be thus enunciated: Given three points and the directions of two asymptotes, to construct the hyperbola.

c. Exercise. — Given four points and the direction of an asymptote, to construct the hyperbola.

a'. Being given five tangents, 1, 2, 3, 4, 5, of a conic, to find a sixth, 6, parallel to one of the given tangents, 1 for example. This is evidently the modification of the general problem (fig. xxxii.) where the point A is at infinity, and in which therefore the diagonal r is parallel to 1 as well as the required tangent 6.



Wherefore having drawn (fig. xxxv.) the diagonal p , draw r parallel to 1, cutting p in P . Through points (2, 3) and P draw q , giving the point X on 5. Through X draw 6 parallel to 1, 6 is the tangent required.

b'. Being given (fig. xxxvi.) five tangents, 1, 2, 3, 4, ∞ , of which one, viz. ∞ , is at infinity, to construct a sixth, 6, through a given point A on one of them, as on 1, or, in other words, Given four tangents of a parabola, to construct a fifth through a given point on one of them.

| | | | | | |
|--------|--------|--------|--------|--------|------------------|
| Sides | 1 | 2 | 3 | 4 | ∞ |
| Points | (6, 1) | (1, 2) | (2, 3) | (3, 4) | (4, ∞) |
| | | | | | (∞ , 6). |

The side 4 thus meets one extremity of $\overset{\infty}{5}$ and 6 the other, whence the diagonal p , through the points (1, 2) and (4, $\overset{\infty}{5}$), must be parallel to 4; the diagonal r goes through the points (3, 4) and A , cutting p in P , whence through (2, 3) and P draw the diagonal q . This diagonal q meets $\overset{\infty}{5}$ in the point $\overset{\infty}{X}$ at infinity, and the required tangent through A goes likewise through $\overset{\infty}{X}$, whence 6 is parallel to q , wherefore, through A draw 6 parallel to q , 6 is the tangent required.

By varying the point A we solve the following problem. To construct an indefinite number of tangents to a parabola determined by four tangents.

Exercise.—Being given four tangents, to construct a tangent parallel to a given straight.

Fig. XXXVI.

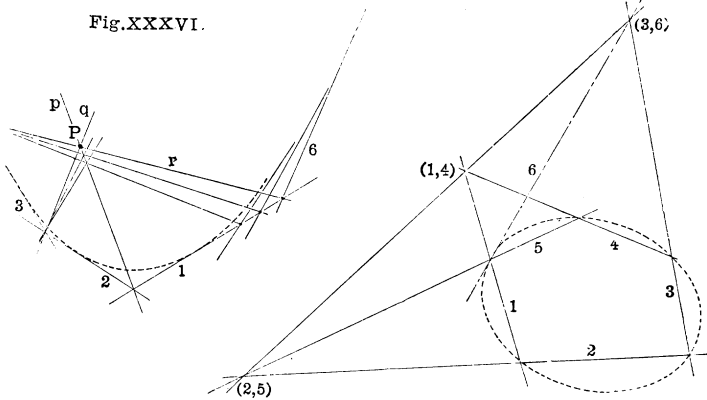


Fig. XXXVII.

L. *One, Two, and Three Sides of PASCAL'S and BRIANCHON'S Theorems becoming infinitely small.*

a. When one side of the Pascal hexagon becomes infinitely small, we have a pentagon, and the infinitely small side assumes the direction of the tangent, whence we can solve the problems:

i. *Given Five Points of a Conic Section, to construct a Tangent through any one of them.*—Join the five points by lines, 1, 2, 3, 4, 5 (fig. xxxvii.), and let (1, 5) be the point through which the tangent is to be drawn. Then extend 1 and 4, 2 and 5, till they

meet in points (1, 4) and (2, 5). Through these two points draw a line, it is the Pascal line, and, where 3 intersects it, is a point in the tangent 6 to the conic at point (1, 5). In the same manner may the tangents belonging to the other points be constructed.

Exercises.—*a. One of the Five Points at Infinity.*—Given four points of a hyperbola and the direction of an asymptote, to construct the asymptote.

b. Two of the Points at Infinity.—Given three points of a hyperbola and the direction of its asymptotes, to construct the tangent at one of the given points, also to construct the asymptotes.

ii. Given Four Points of a Conic and a Tangent at one of them, to construct the Conic by Points.—The analogy with *xlvi. a* and *fig. xxx.* will be sufficient to suggest to the student the construction of this problem.

a'. When one side of the Brianchon hexagon becomes infinitely small, we have a circumscribed pentagon, the point of tangency of the first and the infinitely small sixth side coinciding. The following theorem does not require formal demonstration. If a pentagon is circumscribed to a conic, and the diagonals joining its summits 1 and 4, 2 and 5 be drawn, intersecting in *P*, and if summit 3 and *P* be joined, $\overline{3, P}$ will intersect the side $\overline{1, 5}$ in the point of tangency, whence the problem, Being given five tangents of a conic, to find the point of tangency of one of them; causing the numbers of the summits to revolve round the angles of the pentagon, we can obtain the five points of tangency.

Exercises.—1. Being given four tangents of a parabola, to find their points of contact and their point at infinity.

2. Given four tangents and a point of contact, to construct the conic by tangents. The above theorem and analogy, with *xlix.* and *fig. xxxi.*, will be sufficient to enable the student to construct this problem.

3. Given three tangents and an asymptote of a hyperbola, to construct the hyperbola.

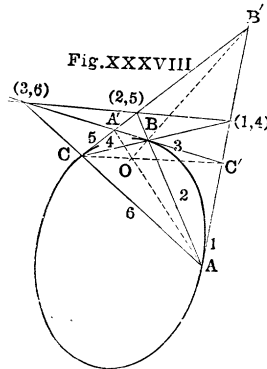
b and *b'*. When two sides of the Pascal and Brianchon hexagons are infinitely small. We have the properties of the quadrilateral already arrived at in Section IV. by another route.

c. When three sides of the Pascal hexagon are infinitely small, fig. xxxviii. In this case sides 1, 3, 5 become tangents, and we obtain the following theorem :

If a triangle, 2, 3, 4, is inscribed in a conic,

| | | |
|-----------|-----------|-----------|
| tangent 1 | side 2 | tangent 3 |
| and | and | and |
| side 4 | tangent 5 | side 6 |

intersect in three points on a straight line, or, The tangents *A*, *B*, *C* at the summits intersect the opposite sides *a*, *b*, *c* in a



straight line, whence the problem, Given three points of a conic, *A*, *B*, *C*, and the tangents at *A* and *B*, to construct the tangent at *C*.

Exercises.—1. Given two points *A* and *B* of a hyperbola, the tangents at these points and the direction of an asymptote, to construct the asymptote.

2. Given an asymptote, a point *A* of a hyperbola, a tangent at *A*, and the direction of the other asymptote, to construct that asymptote.

Theorem.—The triangle inscribed *A*, *B*, *C*, or 2, 4, 6, fig. xxxviii., and the triangle circumscribed *A'*, *B'*, *C'*, or 1, 3, 5,

possess the property that their sides 1 and 4, 2 and 5, 3 and 6 meet in a straight line. They are therefore homologous triangles (x.), and consequently the lines which join the summits cut in a point O . We may enunciate this as follows. If a triangle is circumscribed to a conic, the straights which join the summits to the points of contact of the opposite sides concur in a point, whence the problem, Being given three tangents of a conic and two points of contact, to find the third.

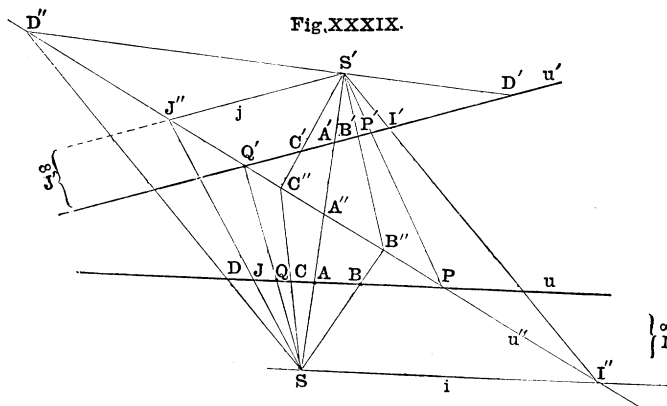
Exercises.—1. Given two tangents, the asymptote of a hyperbola, and the point of contact of one of the tangents, to find the other.

2. Given two asymptotes and a tangent, to construct its point of contact.

3. Given two tangents of a parabola with their points of contact, to construct the direction of the straights which concur at infinity.

4. Given two tangents to a parabola, the point of contact of one of them and the point at infinity, to construct the point of contact of the other tangent.

Section VI.—Projectivity of Points and Rays.



LI. Three Points, A, B, C , on a Line u , fig. xxxix., corresponding to Three Points, A', B', C' , on a Line u' , being both given, we are able to find the corresponding Point D' on u' to any

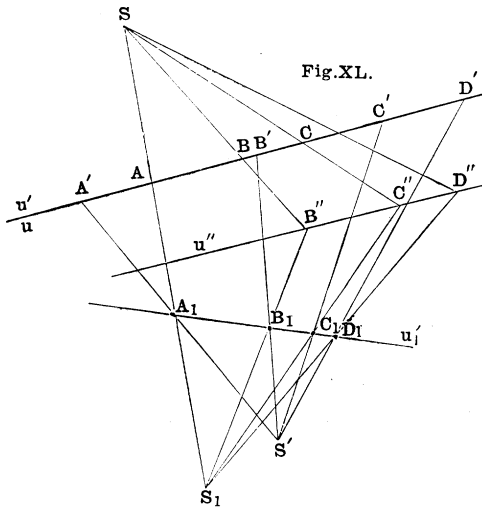
point D on u .—Join any two corresponding points, as A, A' , and on the line thus passing through AA' , take arbitrarily any two points, S, S' . Draw the rays $SB, SC; S'B', S'C'$, and through their points of intersection, B'', C'' , draw the line u'' , and the point A'' is where AA' cuts u'' , and any ray SD will give D'' and $S'D''$ will give D .

In this manner we obtain a row of points u'' , in *perspective* at once with u and u' . The points u are *projective* (not perspective) with u' , and have the same double or anharmonic ratio, for they have each the same ratio as the points u'' , but the two pencils of rays are perspective, not projective.

Let P be the point where u'' cuts u , P' is its correspondent on u' . Let Q' be the point where u'' cuts u' , Q is its correspondent on u .

I' is the correspondent of I at infinity.

J is the correspondent of J' at infinity.



LII. The Solution of the above problem (li.) is required when u and u' are superimposed, that is when A, B, C , and $A'B'C'$ are found upon the same straight, fig. xl.—In this case we must project u' from an arbitrary centre S' upon a straight u_1 , then operate as in xlvii. upon the points $u = A, B, C$, and $u_1 = A_1, B_1, C_1$.

LIII. *Three Rays, a, b, c , of one Pencil U corresponding to Three Rays, a', b', c' , of another Pencil U' being both given, we are able to find another Ray d' of the Second Pencil U' to a given ray d of the First Pencil U , fig. xli.*—Through a point common to two corresponding rays (a, a') for example, lead arbitrarily two transversals s and s' . Let b'' be the straight joining the points (b, s) and (b', s'), c'' the straight joining the points (c, s) and (c', s'). The operations necessary to pass from the pencil $U(a, b, c)$ to the pencil $U'(a', b', c')$ are: i. the section through s ; ii. the focus U'' where b'' and c'' intersect; iii. the section through s' ; iv. the focus U' , wherefore the same operations will conduct to the ray d' corresponding to any given ray d .

In this manner we have obtained a pencil of rays U'' in perspective with U as well as with U' , wherefore the pencils U and U' are projective, but the rows of points are in perspective.

Let p be the ray joining U and U'' , p' is its correspondent from U' meeting p on the line s' .

Let q' be the ray joining U' and U'' , q is its correspondent from U meeting q' on the line s .

LIV. *United Points in Projective Rows of Points.*—The construction of art. xlviii. becomes modified when two corresponding points A, A' coincide, fig. xlii. Leading u , through A , the row of points u , is in perspective with u , wherefore, having projected u' from an arbitrary centre S' upon u , if BR, CC' cut in S it will suffice to project u from S upon u , and finally u , from S' upon u' .

The two rows u, u' of superimposed points have a second united point M at the intersection of the united straight u or u' with the ray SS' , wherefore, if the ray SS' (fig. xliii.) pass through the point (u, u) u and u' will have one united point only.

LV. If we would construct upon a given straight u or u' two rows of projective points, fig. xliii., for which A, A' may be a couple of corresponding points and M the only united point, we must project from an arbitrary point S' the point A' upon a straight u , led through M . Construct the point S common to the straights AA , and $S'M$, then, in order to find the point B'

Fig.XLI.

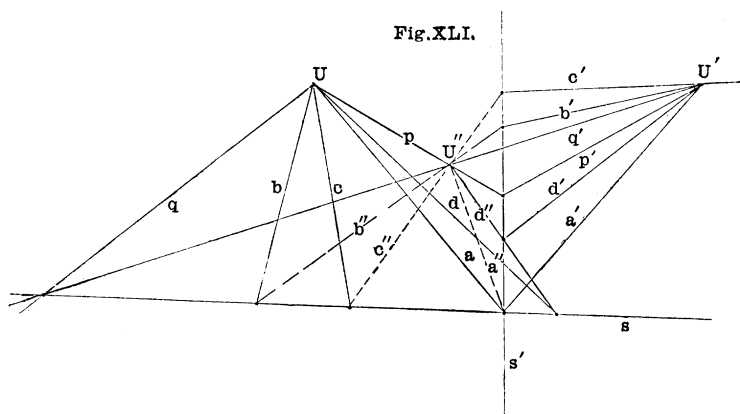


Fig.XLII.

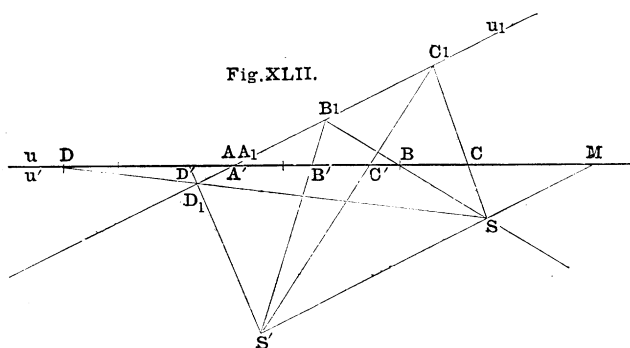
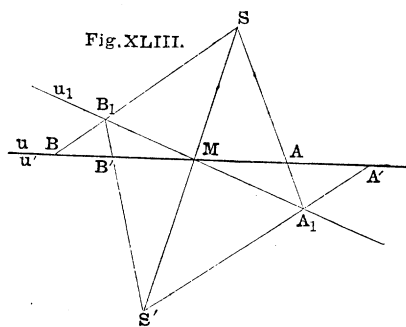


Fig.XLIII.



of u' corresponding to B of u , from S project B to B , and from S' project B to B' .

LVI. Project from two different points U and U' , fig. xlv., by means of two pencils of rays $a, b, c \dots a', b', c', \dots$ upon one given straight the two projective rows of points $u = A, B, C \dots u' = A', B', C'$. Let i, j , be the rays parallel to the given straight from U and U' respectively, and consequently meeting the given straight at infinity, i', j' , their corresponding rays. The points I', J where these last rays cut the given straight are the points which correspond to the point at infinity I or J' , according as we consider the point at infinity as appertaining to the row of points $A, B, C \dots$ or to the row of points $A', B', C' \dots$.

The projectivity of the two corresponding rows gives an equality in the anharmonic ratio, whence (V.), we have

$$JA \cdot I'A' = JB \cdot I'B' = \text{constant.} \quad (1)$$

Let O be the middle point of the segment JI' , and O' the point which corresponds to O regarded as a point of the first row, then, as the above equation subsists for all points,

$$JA \cdot I'A' = JO \cdot I'O', \quad (2)$$

$$\text{or} \quad (OA - OJ) (OA' - OI') = - OJ(OO' - OI'),$$

$$\text{and as} \quad OI' = - OJ,$$

we have thus

$$OA \cdot OA' = OI' \cdot (OA - OA' + OO'). \quad (3)$$

Let us now enquire, if in the two rows of points superimposed on the same ray there are any united points.

Call E such a point and the above equation will subsist, in replacing A and A' by E , whence

$$OE^2 = OI' \cdot OO'. \quad (4)$$

From whence results that, if $OI' \cdot OO'$ is positive, that is, if O is not found between I' and O' , there are two united points E and F , of which O is the middle point, and which divides harmonically the points I' and O' .

If O lies between I' and O' there can be no united points.

If O' coincides with O there is only one united point.

Section VII.—Involution.

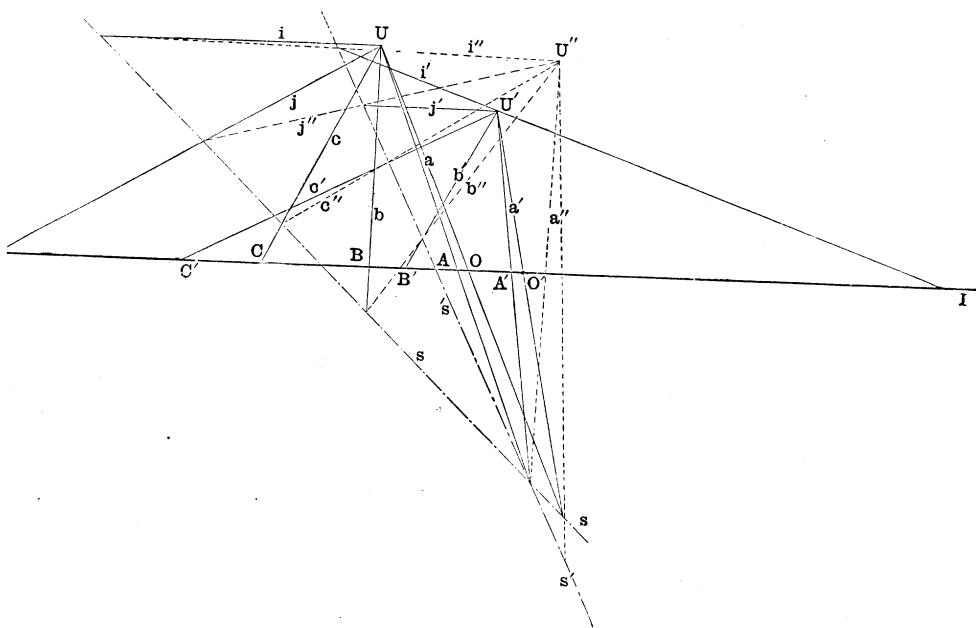
LVII. Let any complete quadrilateral $QRST$, fig. xlv., $QRST$, of which the pairs of opposite sides RT and QS , ST and QR , QT and RS , are cut by any transversal whatever in A and A' , B and B' , C and C' . Take for centres of projection consecutively, Q and S , then

$$\begin{aligned} Q(ACA'B') &= Q(ATPR) = S(ABA'C'') \\ &= A'C'AB \text{ from (xi.)} \end{aligned}$$

whence $ACA'B' = A'C'AB$,

and AA' , BB' , CC' are three couplets of points said to be conjugate to each other, *e.g.* A and A' , C and C' are respectively

Fig. XLIV.



conjugate to each other, because for any two points as A and C may be substituted their conjugates A' and C' without altering the anharmonic ratio.

The three couplets are said to be in involution.

LVIII. Let any complete quadrilateral $qrst$, fig. xlvii., of which the opposite summits rt and qs , st and qr , qt and rs are projected from any arbitrary centre by the rays a and a' , b and b' , c and c' , and let p be the straight which joins qs and rt , then the two pairs of rays $atpr = aca'b'$ for their common section is q . In the same manner $atpr = aba'c'$ for their common section is s , wherefore $aca'b' = aba'c' = a'c'ab$.

The rays aa' , bb' , cc' correspond doubly in the projective groups, wherefore aa' , bb' , cc' are three pairs of conjugate rays in involution.

LIX. When we say that AA' , BB' , CC' form an involution, we are understood to express that A and A' , B and B' , C and C' are conjugate elements, and that any element may be exchanged with its conjugate without altering the anharmonic ratio, so that the two forms

$$AA'BB'CC' \dots$$

$$A'AB'BC'C \dots$$

are projective.

LX. As involution is only a particular case of two projective forms superimposed, it follows that all sections or projections of an involution give a new involution, whence the figure homologous to an involution is likewise an involution.

LXI. *The Centre O of Involution.*—We have seen, art. lvi., when two projective rows of points are superimposed, that, to the point at infinity I or J' , corresponded two finite points (*i.e.* points not at infinity) I' or J , according as we considered it as appertaining to the row u or u' . The distinction in involution is this, that to the point at infinity corresponds one point only, that is, the points I' and J coincide in one point which we shall call O . This point O is therefore the conjugate to the point at infinity, and equation (2) art. lvi. becomes

$$OA \cdot OA' = OB \cdot OB' = \text{constant.}$$

In other words, an involution of points is formed by pairs of points AA' , BB' which possess the property that the

product of their distances from a certain fixed point O of the given straight is constant.

The point O is called the centre of involution.

LXII. Double or United Points in an Involution.—Those points, which in the case of superposition of two forms u and u' have been called united points, are called in the case of involution double points.

LXIII. Existence of Two Double Points in Involution.—If the value in the above equation is positive, that is, if O does not fall between the two conjugate points, there are two double points E and F , such that

$$OE^2 = OF^2 = OA \cdot OA' = OB \cdot OB' = \dots$$

O is therefore the middle of the segment EF and all the groups $EF AA'$, $EF BB'$ are harmonic.

An involution is determined by two pairs of conjugate elements.

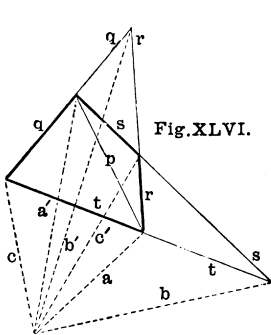


Fig. XLVI.

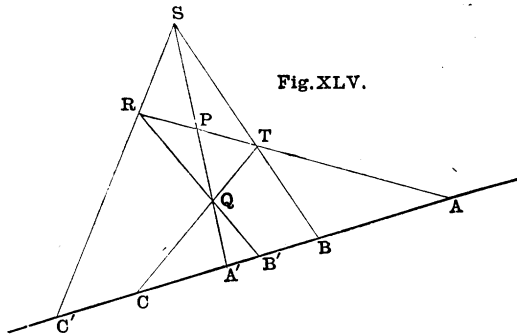
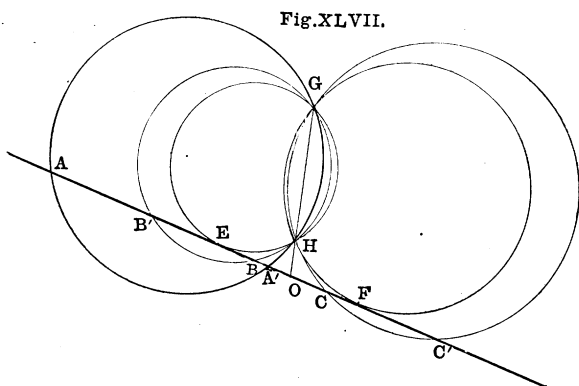


Fig. XLV.

LXIV. Problem.—Let it be required to construct an involution of points upon which are given AA' , BB' . Take any point G , fig. xlvii., without the straight, and describe the circles GAA' , GBB' , cutting in a second point H , and let O be the point where GH encounters the straight, we have, from the well-known property of the circle,

$$OG \cdot OH = OA \cdot OA' = OB \cdot OB' = \dots$$

wherefore the point O is the centre of the involution determined by the two pairs of points AA' , BB' .

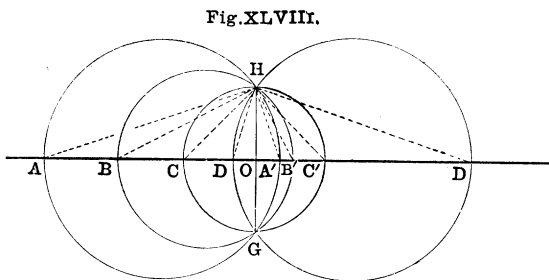


If we describe any other circle through G and H , encountering, say, the given straight in C, C' , we will have

$$OG \cdot OH = OC \cdot OC',$$

wherefore CC' are a pair of points of the involution.

We see further that the two double points E and F of the involution are the points of contact of two circles passing through G and H and touching the given straight. This is named an involution of the first species.



LXV. When the value in equation is negative, *i.e.* when one couple of points divides the other couple, the construction is as follows: Let AA', BB' (fig. xlviii.) be two pairs of points in involution dividing each other. Upon AA' and BB' as diameters describe two circles, cutting each other in the points G and H , necessarily symmetrical to the given line, and cutting in O ,

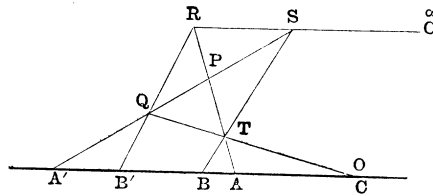
$$OG^2 = OH^2 = AO \cdot OA' = BO \cdot OB' \dots$$

If we describe any other circle through G and H having its centre in the given line, we obtain two other points $C C'$ of the involution.

LXVI. If one of the points as C' of the involution, in art. lvii., is at infinity, its conjugate will be the central point O of the involution, wherefore, in order to find the central point O of the involution of which we have two pairs of conjugate points AA' , BB' , we construct a complete quadrangle (fig. xlix.) in such a manner that two opposite sides pass through A and A' , two other opposite sides through B and B' , and, in order that the fifth side may be parallel to the straight given, the sixth side must pass through O .

LXVII. Let H (fig. xlviii.) be the centre of any number of pairs of rays in involution. We see that the pairs of points on the straight are formed by the revolution of a right angle around H , i.e. AHA , BHB . . . are each a right angle, and O is, as formerly with any pair as A and A' , the point that has its correspondent at infinity, for OHJ' is a right angle meeting the straight in the point at infinity.

Fig XLIX.



LXVIII. Poncelet's method of proof of art. lvii. of the involution of six points in a transversal cutting a quadrangle, and an alternative form of their anharmonic equations. Let the reader form for himself a figure similar to fig. xlv. but having the quadrilateral $QRST$ right-angled. Then it may be considered a projection of that figure, of which the projective properties will be true for both. Then from it the following equations are easily obtained : —

$$\frac{AC \cdot A'C'}{AB \cdot A'B'} = \frac{A'C \cdot A'C'}{A'B \cdot A'B'} \quad \frac{BC \cdot BC'}{BA \cdot BA'} = \frac{B'C \cdot B'C'}{B'A \cdot B'A'}$$

$$\frac{CB \cdot CB'}{CA \cdot CA'} = \frac{C'B \cdot C'B'}{C'A \cdot C'A'}^1$$

These equations shew that in the anharmonic ratios of any four points, the three points ABC are interchangeable with the three points $A'B'C'$, and by means of multiplication give easily the following four equations

$$AC \cdot C'B \cdot B'A' = C'A' \cdot B'C \cdot AB,$$

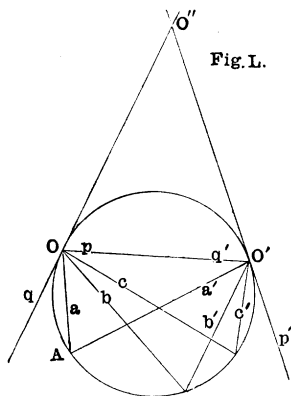
$$C'B' \cdot BA \cdot CA' = BC \cdot B'A' \cdot C'A,$$

$$AC' \cdot CB' \cdot BA' = CA' \cdot C'B \cdot AB',$$

$$C'B' \cdot BA' \cdot CA = BC \cdot B'A \cdot C'A'.$$

These may be put into the following form:—

$$\frac{AC \cdot C'B \cdot B'A'}{C'A' \cdot B'C \cdot AB} = 1.$$



Section VIII.—Forms Projective in the Circle.

LXIX. In two pencils of rays, $O, a, b, c, d, O', a', b', c', d'$ (fig. 1.), proceeding from two O, O' of a circle, the corresponding rays a and a', b and $b' \dots$ intersecting in the circumference of the same circle. These two pencils are projective.

¹ The student can verify the first of these three equations by aid of art. vi., and remembering that $\sin(90^\circ + 45^\circ) = \sin 45^\circ$, he will at once perceive the similar triangles from which the other two are deduced.

For we have seen that the projectivity of two pencils of rays depended solely on the equianharmonism of their angles, and in this case, their corresponding angles being equal, *i.e.* $a'b = a'b'$ (*Euclid*, iii. 21, 22) the equianharmonism of the two pencils is established, whence the projectivity is established.

The ray p joining O and O' considered as pertaining to the pencil O has for its correspondent the tangent p' going through the point O' , and the ray q' joining O and O' considered as pertaining to the pencil O' has for its correspondent the tangent q going through the point O .

For let a mobile point A travelling on the circumference of the circle, connected to O and O' by two mobile rays AO , AO' (or a and a') generate the two pencils, then, when the point A has approached indefinitely near O , the ray AO' will be indefinitely near OO' or q' , and the ray AO will be indefinitely near q or the tangent at O . In the same manner OO' or p will give $A'O'$ or p' tangent at O' .

Corollary. If a pencil of four rays as $O(ABCD)$ is harmonic, $O'(ABCD)$ is likewise harmonic.

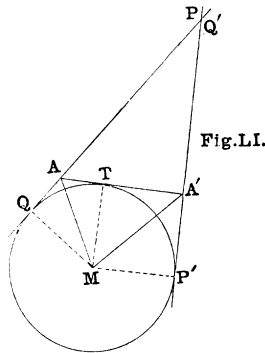


Fig. LI.

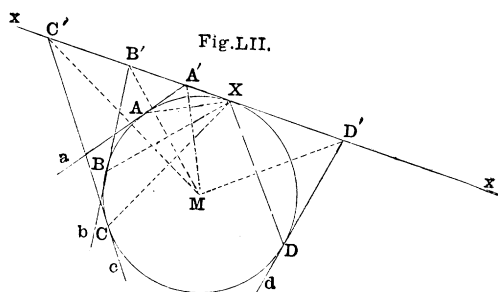
LXX. If (fig. li.) QP , $P'Q'$ are two fixed tangents to a circle, centre M and AA' a tangent mobile between them, the angle AMA' is constant. Let Q , P' , T be the points of contact, we have

$$\angle AMA' = \angle AMT = \frac{1}{2}\angle QMT + \frac{1}{2}\angle TMP' = \frac{1}{2}\angle QMP'.$$

LXXI. The tangent AA' moving between the two fixed tangents the rays MA, MA' generate two projective pencils, and consequently the two points A, A' describe two projective rows of points, wherefore tangents to a circle mark upon two fixed tangents two projective rows of points.

As $\angle AMA' = \frac{1}{2}QMP' = \frac{1}{2}QM'Q' + \frac{1}{2}PMP'$ or P and Q' indicate the same point considered as pertaining to the first or to the second fixed tangent, the points Q and Q', P and P' are correspondents in the two projective rows of points, or in other words the points of contact of two fixed tangents have for their correspondents the point in common of both tangents.

For conceive the mobile tangent AA' to turn round upon the circle, the points A, A' generating two projective rows of points, then when it has reached a position indefinitely near PQ , the point A' is indefinitely near Q' , and A , the corresponding point of the other row, is indefinitely near Q , wherefore the point of contact of a fixed tangent may be considered as the intersection with a tangent infinitely near. In other words :

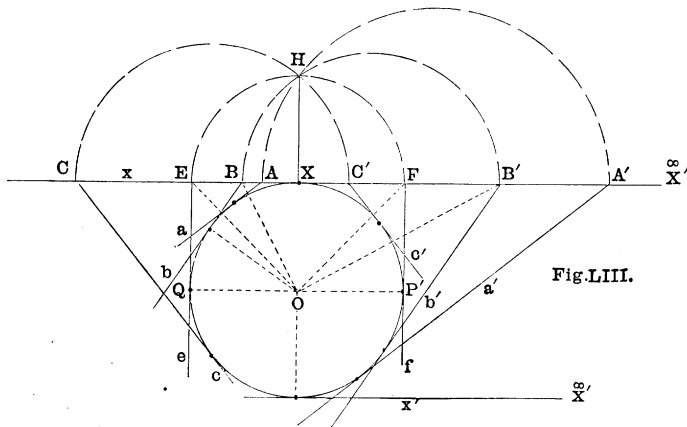


LXXII. Four tangents a, b, c, d , of a circle (fig. lii.) are cut by a fifth tangent in four points $A'B'C'D'$ of which the anharmonic ratio is constant wherever the fifth tangent is placed. Let fig. lii., $A, B, C, D \dots X$ be points of a circle and $a, b, c, d \dots x$ their corresponding tangents. If we project from the centre of the circle the points $A', B', C', D' \dots$ where x is cut by $a, b, c, d \dots$ the projected rays are respectively perpendicular to $XA, XB, XC \dots$ and form consequently a pencil equal to the pencil $X(ABC \dots)$, wherefore the row of points $A', B', C' \dots$ is projective with the pencil $X(A, B, C \dots)$,

or in other words, the row of points determined by an indefinite number of tangents upon an arbitrarily chosen tangent is projective with the pencil of rays which project the points of contact from any point of the circle.

LXXIII. *Case of Involution in the Circle.*—Let fig. liii. represent a circle, centre O , enveloped by pairs of parallel tangents aa' , bb' , cc' , ef determining upon a given tangent x pairs of points $A, A', B, B', C, C' \dots EF$, e and f being perpendicular to x , whence $EX = XF$ $\angle EOF$ being $= \frac{1}{2} \angle QOP'$ (lix.) is a right angle, for the same reason $\angle AOA'$, $\angle BOB'$, $\angle COC'$ are each of them a right angle, whence AA' , BB' , $CC' \dots$ is an involution of the second species, and X' the correspondent of X is the point at infinity, whence X is the centre of the involution.

It is evident from elementary considerations that a ray from the centre of the circle parallel to any pair of tangents, for instance a ray from O parallel to AA' , cuts the tangent x in the centre of the circle $AHA' \dots$



Section IX.—Forms Projective in the Conics.

LXXIV. If we project two pencils of rays from two fixed points O and O' of a conic, fig. liv., to any number of points $A, B, C, D \dots$ of that conic ($O(A, B, C, D \dots)$ and $O'(A, B, C, D \dots)$) form two projective pencils. To the ray

OO' of the first pencil corresponds the tangent at O' , and to the ray $O'O$ of the second pencil corresponds the tangent at O . For this proposition having been proved with respect to the circle, let figure liv. be a circle and such a scheme of rays and its homologous projection into a conic from a centre S of homology. Let the rays in the circle be cut by any transversal t' , then we have the plane $S't'$ cutting the plane a in the line t , and we have a pencil of rays from S , viz. $S(t'a, t'b, t'c \dots t'a', t'b', t'c' \dots)$ cut by two transversals t and t' , giving two rows of points $ta, tb, tc \dots ta', tb', tc' \dots$ and $t'a, t'b, t'c \dots t'a', t'b', t'c' \dots$ in the conic and circle in perspective and consequent four and four equianharmonic, wherefore the rays from O and O' in the conic are equianharmonic with the rays at O and O' in the circle, whence as the two pencils of rays in the circle are projective, the two pencils in the conic are also projective.

LXXV. By similar reasoning the propositions of the preceding section can be extended to the conics.

LXXVI. Article lxix., extended to conics generally, is more simply elucidated by placing the centre of a homology upon a point in the conic, so as to be its own correspondent, as explained in xxv. *a*.

If two pencils of rays traced in the same plane not concentric are projective (not perspective) the locus of the point common to two corresponding rays is a conic, which passes through the centres of these pencils, and the tangents at these points are the rays of the two pencils which correspond to the straight joining their centres. Let O and S (fig. lv.) be the centres of the two pencils of rays, OA_1SA , $OA_2SA_2 \dots$ projective not perspective. The locus of the points $A_1A_2A_3 \dots$ passes through O , because the ray SO of the pencil S and the corresponding ray of the pencil O intersect in O , that is A coincides with O . For the same reason locus of A goes likewise through S . Let o be the ray of the pencil O which corresponds to the ray SO of the pencil S . Describe a circle tangential to o at the point O , and let S' be the point where this circle cuts SO . Let A' be the intersection of the ray OA with the circle, the pencil $S'A'$ will be directly equal to the pencil OA , it will therefore be *projective* to the

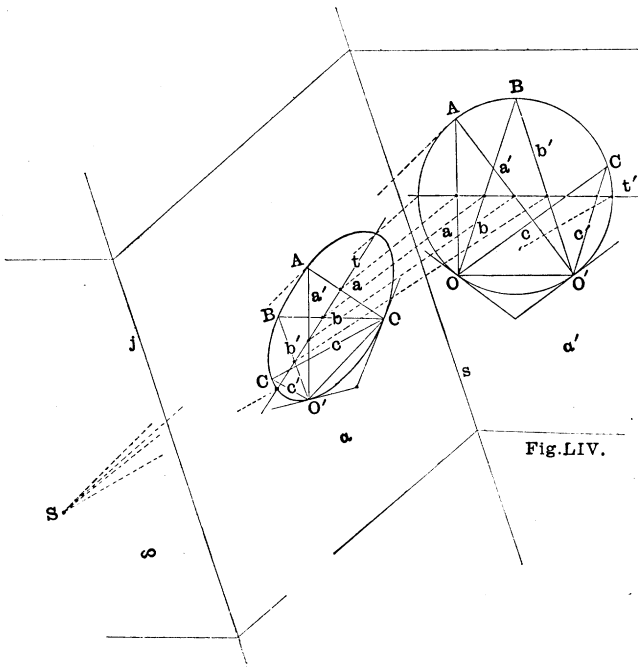


Fig. LIV.

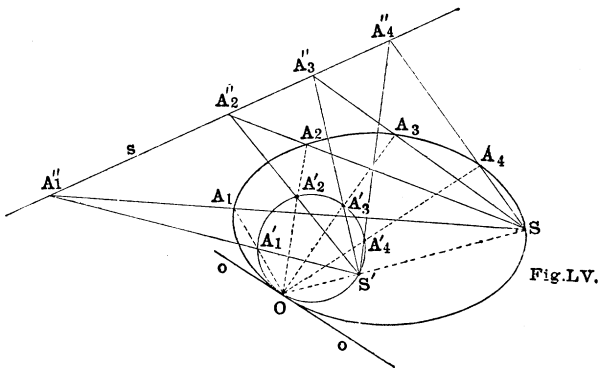


Fig. LV.

pencil SA , and as the ray $S'O$ corresponds to the ray SO , the pencils SA and $S'A'$ are *perspective*, wherefore their corresponding rays will cut in a straight line s . Wherefore in order to construct the locus required, it is sufficient to prolong SA till it meets the straight s in A'' drawn SA'' , and through the point where this last cuts the circle draw a ray from O intersecting SA'' in the locus required.

This process is, we know, that of finding the figure homologous to a circle, for to find the curve homologous to a circle we have s the axis and O the centre of homology, and S, S' two corresponding points.

LXXVII. Article lxxii. is enunciated as follows: If two rows of points in two straights, situated in the same plane but not superimposed, are projective (not perspective), the straights which join the pairs of corresponding points envelope a conic, *i.e.* are tangents to a conic. This conic touches the two given straights at the two points which correspond to their intersection. Let s, s' (fig. lvi.) be the straights upon which are two rows of projective points, A and A', B and $B' \dots$ then the curve enveloped by the straights $AA', BB', CC' \dots$ also by s and s' is a conic.

Let Q and P' be the points corresponding to P and Q' the intersection of s and s' and describe a circle tangent to s at Q and lead to it the tangents $AA'', BB'', CC'' \dots MM''$. These tangents mark upon s'' a row of points $A'', B'', C'' \dots M''$ projective with the row of points s and also with s' . But the point Q has Q' for its correspondent, both in s' and s'' , wherefore the rows on s' and s'' are *perspective*, and the straights $A''A', B''B' \dots M''M'$ concur in a point O . From thence it results that if we determine the point M' of s' which corresponds to a point MM'' of s it will suffice to lead through M a tangent to the circle, to join the point M'' on s'' with O , OM'' cutting s' in M' . Join MM' it is a straight of the envelope. For this construction is precisely that which we adopt in order to find the figure homologous to a circle, O being the centre of homology and s', s'' two corresponding straights on a and a' , wherefore the envelope of all MM' is a conic.

If a pencil of three rays, fig. lviii., diverging from a point O ,

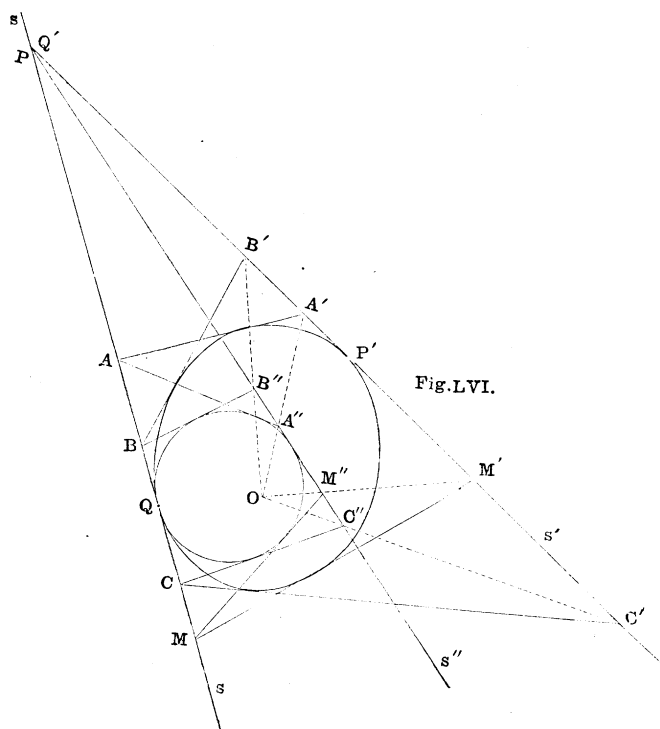


Fig. LVI.

cut the circumference of a conic in AA' , BB' , CC' , the six points thus obtained form a pencil of involution when joined to any point of the curve. Let m be another ray of the pencil O cutting the circumference in MM' , the two pencils $M(A, B, C, C')$ and $M'(A, B, C, C')$ have the same anharmonic ratio (lxix.). But the straights $M'A$ and MA' , $M'B$ and MB' , $M'C$ and MC' , $M'C'$ and MC cut two and two upon the polar of O (xliv.), wherefore this polar being a secant common to the pencil $M'(A, B, C, C')$ which we have been considering, and to the pencil $M(A'B'C'C)$, they have the same anharmonic ratio, and as we have seen

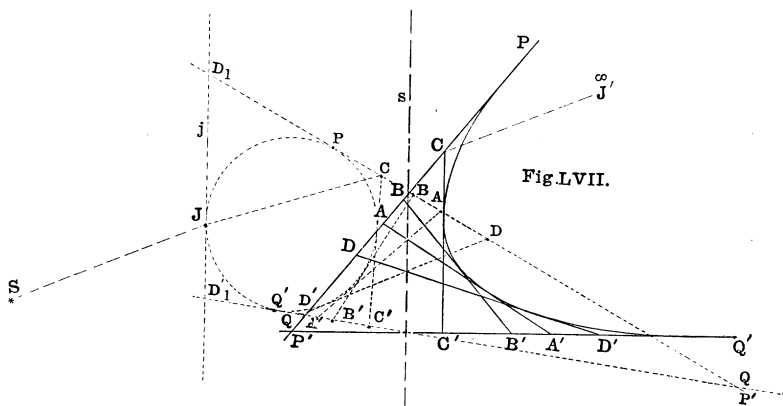
$$M'(ABCC') = M(ABCC'),$$

whence

$$M(A'B'C'C) = M'(ABCC') = M'(ABCT).$$

If M approaches infinitely near C , the rays MC and $M'C'$ become the tangents at M and M' .

LXXVIII. *Extension of lxxvii. to the Parabola.*—This extension leads to the following theorem: Two tangents of a parabola are divided by all other tangents proportionally and conversely. A series of lines dividing two other lines proportionally envelope



a parabola. Let two fixed tangents be cut by others (fig. lvii.) in the points A and A' , B and B' , C and C' Let P , Q' be their points of contact with the curve, then their common point will be designated Q or P' according as we consider it a point in the first or second tangent, then we have generally

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$$

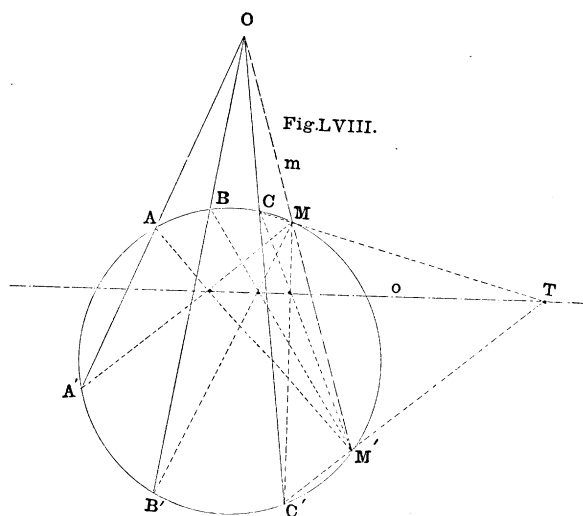
when D and D' are at infinity this becomes

$$\frac{AC}{BC} = \frac{A'C'}{B'C'} \quad \text{or} \quad \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

When AA' approaches infinitely near PQ , A coincides with P and A' with P' . When CC' approaches infinitely near $Q'P$, C coincides with Q , and C' with Q' , and the above equality may be written

$$\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{PQ}{P'Q'}$$

To assist conception, the homologous circle and rays have been given in the figure in faint dotted lines.



LXXIX. DESARGUES' *Theorem in Conics*.—A quadrilateral, $QRST$, fig. lix., being inscribed to a conic, any transversal s determines, by its intersections in PP' of the curve, and in AA' , BB' of the quadrilateral, six points, AA' , BB' , PP' , in involution.

Take Q for the centre of projection and then S , and we see that

$$Q(PBP'A) = S(PA'P'B')$$

and by (x.)

$$(PBP'A) = (P'APB)$$

whence

$$(P'APB) = PA'P'B',$$

whence the six points AA' , BB' , PP' are seen to be in involution.

LXXX. *Reciprocal to DESARGUES' Theorem in Conics*.—A quadrilateral, $qrst$, fig. lx., being circumscribed to a conic, any pencil of rays, aa' , bb' , from an arbitrary focus S to the opposite summits (qt) (rs) and (qr) (ts) of the quadrilateral, determine, with two rays pp' touching the conic, three pairs of rays in involution.

Let q be considered one transversal and s another, then

$$S(PRT) = U(PR, T),$$

whence

$$pba \text{ is projective with } pa'b'.$$

Insert a fourth term,

$pbp'a$ is projective with $pa'p'b'$,

$pa'p'b'$ is projective with $p'b'pa'$,

whence

$pbp'a$ is projective with $p'b'pa'$,

and the six rays pp' , bb' , aa' are in involution.

The two theorems given above are perhaps the most powerful instruments in the geometrical investigation of conic sections, and with the example of the employment of the first, where no points are at infinity, we will close our introduction to projective geometry.

LXXXI. *Problem*.—Given five points of a conic section, $PQRST$, fig. lxi., of a conic, to find an indefinite number of sixth points P' . Having led a transversal s arbitrarily through P , construct the new quadrilateral $Q'R'S'T'$, of which the diagonal $R'T'$ passes through P , and of which the pairs of opposite sides $Q'R'$ and $S'T'$ and $R'S'$ concur upon s with the pairs of similarly named opposite sides of the first quadrilateral $QRST$ in the points B and B' A and A' the second diagonal $Q'S'$ will cut the transversal in the point P' required.

For the points P , P' of the conic are, when regarded as belonging to the second quadrilateral, the points AA' of fig. xlv., likewise in involution with CC' , BB' .

LXXXII. PONCELET'S *Demonstration of above Theorems*.—Let the conic, with its inscribed quadrilateral, be projected into a circle and inscribed rectangle, and cut by the projection of the transversals, then, from the well-known property of the circle, we have

$$\frac{BP' \cdot BP}{B'P' \cdot B'P} = \frac{BQ \cdot BR}{B'T \cdot B'S}$$

But because of similar triangles we have also

$$\frac{BQ}{B'T} = \frac{BA}{B'A}, \quad \frac{BR}{B'S} = \frac{BA'}{B'A'}$$

whence

$$\frac{BP' \cdot BP}{B'P' \cdot B'P} = \frac{BA \cdot BA'}{B'A \cdot B'A'}$$

a relation which is evidently the same as the second of the three primary relations of art. lxviii. Similarly the other two might be deduced.

Fig.LIX.

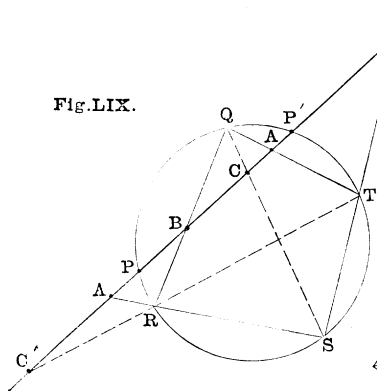


Fig.LX.

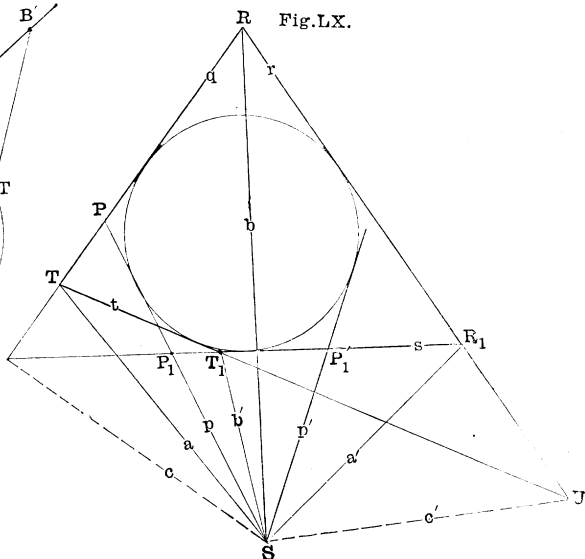
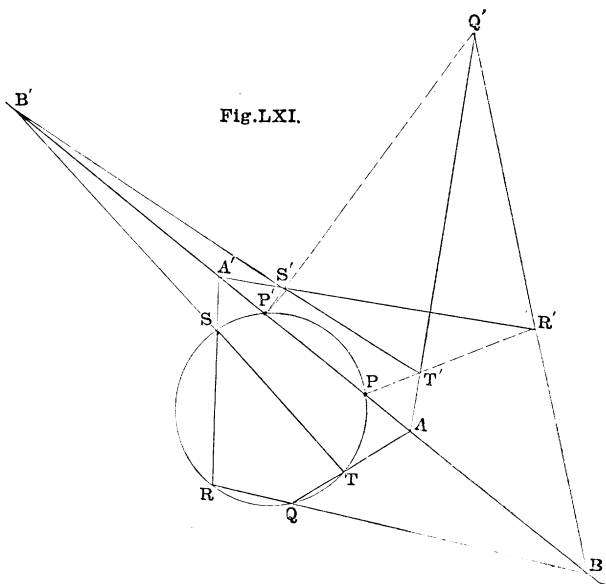


Fig.LXI.



LONDON:
R. CLAY, SONS, AND TAYLOR,
BREAD STREET HILL.

MESSRS. MACMILLAN & CO.'S PUBLICATIONS.

SIMPLE PRACTICAL METHODS OF CALCULATING STRAINS ON GIRDERS, ARCHES, AND TRUSSES. With a Supplementary Essay on Economy in Suspension Bridges. By G. W. YOUNG, Associate of King's College, London, Member of the Institution of Civil Engineers. With Diagrams. 8vo. 7s. 6d.

THE STEAM ENGINE AND ITS INVENTORS: A HISTORICAL SKETCH. By ROBERT L. GALLOWAY, Mining Engineer. With numerous Illustrations. Crown 8vo. 10s. 6d.

EXPERIMENTAL MECHANICS. Lectures delivered at the Royal College of Science for Ireland. By R. S. BALL, M.A., Professor of Applied Mathematics and Mechanics. Illustrated. Cheaper Issue. Royal 8vo. 10s. 6d.

THE KINEMATICS OF MACHINERY; OUTLINES OF A THEORY OF MACHINES. By Professor F. REULEAUX. Translated and Edited by A. B. W. KENNEDY, C.E., Professor of Civil and Mechanical Engineering in University College, London. With 450 Illustrations. Medium 8vo. 21s.

THE MECHANICAL THEORY OF HEAT. By R. CLAUSIUS. Translated by WALTER R. BROWNE, M.A., late Fellow of Trinity College, Cambridge. Crown 8vo. 10s. 6d.

THE APPLICATION OF ELECTRICITY TO RAILWAY WORKING. By W. E. LANGDON, Engineering Superintendent of Post-Office Telegraphs. With numerous Illustrations. Extra fcap. 8vo. 4s. 6d.

AN ELEMENTARY TREATISE ON HEAT IN RELATION TO STEAM AND THE STEAM ENGINE. By G. SHANN, M.A. With numerous Illustrations. Crown 8vo. 4s. 6d.

ELECTRICITY. An Introduction to the Theory of Electricity. With numerous Examples. By LINNÆUS CUMMING, M.A., Assistant Master at Rugby School. Crown 8vo. 8s. 6d.

BLOWPIPE ANALYSIS. By J. LANDAUER. Authorised English Edition by JAMES TAYLOR and WILLIAM E. KAY, of the Owens College, Manchester. With Illustrations. Extra fcap. 8vo. 4s. 6d.

METALS AND THEIR CHIEF INDUSTRIAL APPLICATIONS. By CHARLES R. ALDER WRIGHT, D.Sc., &c. Lecturer on Chemistry in St. Mary's Hospital School. Extra fcap. 8vo. 3s. 6d.

MACMILLAN AND CO., LONDON.

MESSRS. MACMILLAN & CO.'S PUBLICATIONS (*cont.*)

ELEMENTS OF DYNAMICS. An Introduction to the Study of Motion and Rest in Solid and Fluid Bodies. Part I. Kinematic. By W. K. CLIFFORD, F.R.S., late Fellow and Assistant Tutor of Trinity College, Cambridge; Professor of Applied Mathematics and Mechanics at University College, London. Crown 8vo. 7s. 6d.

AN ELEMENTARY TREATISE ON STEAM. By JOHN PERRY, Whitworth Scholar, Fellow of the Chemical Society, Lecturer in Physics at Clifton College. With numerous Woodcuts, Numerical Examples, and Exercises. *New Edition.* 18mo. 4s. 6d.

MECHANICS FOR BEGINNERS. With numerous Examples. *New Edition.* By ISAAC TODHUNTER, M.A., F.R.S. 18mo. 4s. 6d.; Key, 6s. 6d.

ELEMENTARY APPLIED MECHANICS. Being the simpler and more practical Cases of Stress and Strain wrought out individually from first principles by means of Elementary Mathematics. By T. ALEXANDER, C.E., Professor of Civil Engineering in the Imperial College of Engineering, Tokei, Japan. Crown 8vo. 4s. 6d.

THE FORCES OF NATURE. A popular Introduction to the Study of Physical Phenomena. By AMÉDÉE GUILLEMIN. Translated from the French by Mrs. NORMAN LOCKYER; and Edited, with Additions and Notes, by J. NORMAN LOCKYER, F.R.S. Illustrated by 455 Woodcuts. Royal 8vo. cloth extra. 21s.

THE APPLICATIONS OF PHYSICAL FORCES. By AMÉDÉE GUILLEMIN. Translated from the French by Mrs. NORMAN LOCKYER; and Edited, with Additions and Notes, by J. NORMAN LOCKYER, F.R.S. With Coloured Plates and Illustrations. Royal 8vo. Cheaper Issue. 21s.

SCIENCE LECTURES AT SOUTH KENSINGTON. With Illustrations. Vol. I. Containing Lectures by Captain ABNEY, R.E., F.R.S.; Professor STOKES; Professor A. B. W. KENNEDY, C.E.; F. J. BRAMWELL, C.E., F.R.S.; Professor F. FORBES; H. C. SORBY, F.R.S.; J. T. BOTTOMLEY, F.R.S.E.; S. H. VINES, B.Sc.; Professor CAREY FOSTER. Crown 8vo. 6s.

Vol. II. Containing Lectures by W. SPOTTISWOODE, P.R.S.; Professor FORBES; H. W. CHISHOLM; Professor T. F. PIGOT; W. FROUDE, LL.D., F.R.S.; Dr. SIEMENS; Professor BARRETT; Dr. BURDON-SANDERSON; Dr. LAUDER BRUNTON, F.R.S.; Professor MCLEOD; Professor ROSCOE, F.R.S. With Illustrations. Crown 8vo. 6s.

HOW TO DRAW A STRAIGHT LINE. A Lecture on Linkages. By A. B. KEMP, B.A., of the Inner Temple; Member of the Council of the Mathematical Society; and late Scholar of Trinity College, Cambridge. With numerous Illustrations. Crown 8vo. 1s. 6d. (*Nature Series.*)

A TREATISE ON APPLIED MECHANICS. By JAMES COTTERILL, M.A., F.R.S., Professor of Applied Mechanics at the Royal Naval College, Greenwich. With Illustrations. 8vo. [*In preparation.*]

AN ELEMENTARY TREATISE ON MECHANICS. By A. B. KENNEDY, C.E., Professor of Applied Mechanics in University College, London. With Illustrations. [*In the press.*]

MACMILLAN AND CO., LONDON.